Orthogonal Complements

DEFINITION: If a vector $\mathbf{z}$ is orthogonal to every vector in a subspace W of $\mathbb{R}^n$, then $\mathbf{z}$ is said to be **orthogonal to** W.

EXAMPLE: Let $\mathbf{z} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$ and

$$W = \left\{ \begin{bmatrix} 6 \\ -4 \end{bmatrix} t : \ t \in \mathbb{R} \right\}$$

be a subspace of $\mathbb{R}^2$. Then $\mathbf{z}$ is orthogonal to every vector in W, since

$$\mathbf{z} \cdot \left(\begin{bmatrix} 6 \\ -4 \end{bmatrix} t \right) = \left(\begin{bmatrix} 2 \\ 3 \end{bmatrix} \cdot \begin{bmatrix} 6 \\ -4 \end{bmatrix} \right) t = 0 \cdot t = 0$$

EXAMPLE: Let $\mathbf{z} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ and

$$W = \left\{ \begin{bmatrix} -1 \\ 5 \\ -3 \end{bmatrix} t_1 + \begin{bmatrix} 7 \\ 4 \\ -5 \end{bmatrix} t_2 : \ t_1, t_2 \in \mathbb{R} \right\}$$

be a subspace of $\mathbb{R}^3$. Then $\mathbf{z}$ is orthogonal to every vector in W, since

$$\mathbf{z} \cdot \left(\begin{bmatrix} -1 \\ 5 \\ -3 \end{bmatrix} t_1 + \begin{bmatrix} 7 \\ 4 \\ -5 \end{bmatrix} t_2 \right)$$

$$= \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \cdot \begin{bmatrix} -1 \\ 5 \\ -3 \end{bmatrix} t_1 + \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \cdot \begin{bmatrix} 7 \\ 4 \\ -5 \end{bmatrix} t_2 = 0 \cdot t_1 + 0 \cdot t_2 = 0$$

DEFINITION: The set of all vectors $\mathbf{z}$ that are orthogonal to W is called the **orthogonal complement** of W and is denoted by $W^{\perp}$.

EXAMPLE: Let

$$H = \left\{ \begin{bmatrix} 2 \\ 3 \end{bmatrix} s : \ s \in \mathbb{R} \right\}$$

and

$$W = \left\{ \begin{bmatrix} 6 \\ -4 \end{bmatrix} t : \ t \in \mathbb{R} \right\}$$

be subspaces of $\mathbb{R}^2$. Then every vector in H is orthogonal to every vector in W, since

$$\left(\begin{bmatrix} 2 \\ 3 \end{bmatrix} s \right) \cdot \left(\begin{bmatrix} 6 \\ -4 \end{bmatrix} t \right) = \left(\begin{bmatrix} 2 \\ 3 \end{bmatrix} \cdot \begin{bmatrix} 6 \\ -4 \end{bmatrix} \right) st = 0.$$

Moreover, one can show that there are no other vectors in $\mathbb{R}^2$ which are orthogonal to every vector in W. Therefore $H = W^{\perp}$.

EXAMPLE: Let L_1 be a line through the origin in $\mathbb{R}^2$, and let L_2 be the line through the origin and perpendicular to L_1. Then each vector on L_1 is orthogonal to every vector in L_2. Moreover, one can show that there are no other vectors in $\mathbb{R}^2$ which are orthogonal to every vector in L_1. Therefore

$$L_1 = L_2^{\perp}$$

Also, for the same reason we have

$$L_2 = L_1^{\perp}$$

EXAMPLE: Let $H = \left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} s : \ s \in \mathbb{R} \right\}$ and

$$W = \left\{ \begin{bmatrix} -1 \\ 5 \\ -3 \end{bmatrix} t_1 + \begin{bmatrix} 7 \\ 4 \\ -5 \end{bmatrix} t_2 : \ t_1, t_2 \in \mathbb{R} \right\}$$

be subspaces of $\mathbb{R}^3$. Then every vector in H is orthogonal to every vector in W, since

$$\left(\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} s \right) \cdot \left(\begin{bmatrix} -1 \\ 5 \\ -3 \end{bmatrix} t_1 + \begin{bmatrix} 7 \\ 4 \\ -5 \end{bmatrix} t_2 \right)$$

$$= \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \cdot \begin{bmatrix} -1 \\ 5 \\ -3 \end{bmatrix} st_1 + \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \cdot \begin{bmatrix} 7 \\ 4 \\ -5 \end{bmatrix} st_2 = 0 \cdot st_1 + 0 \cdot st_2 = 0$$

Moreover, one can show that there are no other vectors in $\mathbb{R}^3$ which are orthogonal to every vector in W. Therefore $H = W^{\perp}$.

EXAMPLE: Let W be a plane through the origin in $\mathbb{R}^3$, and let L be the line through the origin and perpendicular to W. Then each vector on L is orthogonal to every vector $\mathbf{z}$ in W. Moreover, one can show that there are no other vectors in $\mathbb{R}^3$ which are orthogonal to every vector in W. Therefore

$$L = W^{\perp}$$

Also, for the same reason we have

$$W = L^{\perp}$$

THEOREM 6.10: If W is a subspace of $\mathbb{R}^n$, then $\mathbf{v} \in W^{\perp}$ if and only if $\mathbf{v}$ is orthogonal to every vector in a spanning set for W.

THEOREM 6.11: Let W be a subspace of $\mathbb{R}^n$. Then $W^{\perp}$ is a subspace of $\mathbb{R}^n$, and $W \cap W^{\perp} = \{\mathbf{0}\}$.

THEOREM 6.12: Let W be a subspace of $\mathbb{R}^n$. Let $\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ be an orthogonal basis for W contained in an orthogonal basis $\{\mathbf{v}_1, \ldots, \mathbf{v}_k, \mathbf{v}_{k+1}, \ldots, \mathbf{v}_n\}$ for $\mathbb{R}^n$. Then $\{\mathbf{v}_{k+1}, \ldots, \mathbf{v}_n\}$ is an orthogonal basis for $W^{\perp}$.

COROLLARY 6.13: Let W be a subspace of $\mathbb{R}^n$. Then

$$\dim(W) + \dim(W^{\perp}) = n = \dim(\mathbb{R}^n)$$

An Orthogonal Projection

THEOREM 6.15 (Projection Theorem): Let W be a subspace of $\mathbb{R}^n$. Then each $\mathbf{y}$ in $\mathbb{R}^n$ can be written uniquely in the form

$$\mathbf{y} = \hat{\mathbf{y}} + \mathbf{z}$$

where $\hat{\mathbf{y}}$ is in W and $\mathbf{z}$ is in $W^{\perp}$. In fact, if $\{\mathbf{u}_1, \ldots, \mathbf{u}_p\}$ is any orthogonal basis of W, then

$$\hat{\mathbf{y}} = \frac{\mathbf{y} \cdot \mathbf{u}_1}{\mathbf{u}_1 \cdot \mathbf{u}_1}\mathbf{u}_1 + \ldots + \frac{\mathbf{y} \cdot \mathbf{u}_p}{\mathbf{u}_p \cdot \mathbf{u}_p}\mathbf{u}_p$$

and $\mathbf{z} = \mathbf{y} - \hat{\mathbf{y}}$

DEFINITION: The vector $\hat{\mathbf{y}}$ is called the **orthogonal projection** of $\mathbf{y}$ onto W and written as

$$\text{proj}_W \mathbf{y}$$

EXAMPLE: Let

$$\mathbf{u}_1 = \begin{bmatrix} 2 \\ 5 \\ -1 \end{bmatrix}, \quad \mathbf{u}_2 = \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}, \quad \text{and} \quad \mathbf{y} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

(i) Find the orthogonal projection of $\mathbf{y}$ onto $W = \text{Span}\{\mathbf{u}_1, \mathbf{u}_2\}$;

(ii) Write $\mathbf{y}$ as the sum of a vector in W and a vector orthogonal to W.

Solution:

(i) By the Theorem above, the orthogonal projection of $\mathbf{y}$ onto W is

$$\hat{\mathbf{y}} = \frac{\mathbf{y} \cdot \mathbf{u}_1}{\mathbf{u}_1 \cdot \mathbf{u}_1}\mathbf{u}_1 + \frac{\mathbf{y} \cdot \mathbf{u}_2}{\mathbf{u}_2 \cdot \mathbf{u}_2}\mathbf{u}_2$$

$$= \frac{9}{30}\begin{bmatrix} 2 \\ 5 \\ -1 \end{bmatrix} + \frac{3}{6}\begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -2/5 \\ 2 \\ 1/5 \end{bmatrix}$$

(ii) By the Theorem above we have $\mathbf{z} = \mathbf{y} - \hat{\mathbf{y}}$, therefore

$$\mathbf{z} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} - \begin{bmatrix} -2/5 \\ 2 \\ 1/5 \end{bmatrix} = \begin{bmatrix} 7/5 \\ 0 \\ 14/5 \end{bmatrix}$$

So,

$$\mathbf{y} = \begin{bmatrix} -2/5 \\ 2 \\ 1/5 \end{bmatrix} + \begin{bmatrix} 7/5 \\ 0 \\ 14/5 \end{bmatrix}$$

Application: Distance from a Point to a Subspace

DEFINITION: Let W be a subspace of $\mathbb{R}^n$, and assume all vectors in W have initial point at the origin. Let P be any point in n-dimensional space. Then the **minimum distance** from P to W is the shortest distance between P and the terminal point of any vector in W.

THEOREM 6.17: Let W be a subspace of $\mathbb{R}^n$, and let P be a point in n-dimensional space. If $\mathbf{v}$ is the vector from the origin to P, then the minimum distance from P to W is $\|\mathbf{v} - \text{proj}_W \mathbf{v}\|$.