Isomorphism

Definition Let $L\colon \mathcal{V} \to \mathcal{W}$ be a linear transformation. Then L is an **invertible linear transformation** if and only if there is a function $M\colon \mathcal{W} \to \mathcal{V}$ such that $(M \circ L)(\mathbf{v}) = \mathbf{v}$, for all $\mathbf{v} \in \mathcal{V}$, and $(L \circ M)(\mathbf{w}) = \mathbf{w}$, for all $\mathbf{w} \in \mathcal{W}$. Such a function M is called an **inverse** of L.

Definition A linear transformation $L\colon \mathcal{V} \to \mathcal{W}$ that is both one-to-one and onto is called an **isomorphism** from $\mathcal{V}$ to $\mathcal{W}$.

Theorem 5.14 Let $L\colon \mathcal{V} \to \mathcal{W}$ be a linear transformation. Then L is an isomorphism if and only if L is an invertible linear transformation. Moreover, if L is invertible, then L^{-1} is also a linear transformation.

Example 1

Recall the rotation linear operator $L\colon \mathbb{R}^2 \to \mathbb{R}^2$ with

$$L\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$$

given in Example 9 in Section 5.1. In Example 1 in Section 5.4, we proved that L is both one-to-one and onto. Hence, L is an isomorphism and has an inverse, L^{-1}. Because L represents a *counterclockwise* rotation of vectors through the angle θ, then L^{-1} must represent a *clockwise* rotation through the angle θ, as we saw in Example 1 of Section 5.4. Equivalently, L^{-1} can be thought of as a *counterclockwise* rotation through the angle $-\theta$. Thus,

$$L^{-1}\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} \cos(-\theta) & -\sin(-\theta) \\ \sin(-\theta) & \cos(-\theta) \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}.$$

Of course, L^{-1} is also an isomorphism. ■

Theorem 5.15 Let $\mathcal{V}$ and $\mathcal{W}$ both be nontrivial finite dimensional vector spaces with ordered bases B and C, respectively, and let $L\colon \mathcal{V} \to \mathcal{W}$ be a linear transformation. Then L is an isomorphism if and only if the matrix representation $\mathbf{A}_{BC}$ for L with respect to B and C is nonsingular.

Theorem 5.16 Suppose $L\colon \mathcal{V} \to \mathcal{W}$ is an isomorphism. Let S span $\mathcal{V}$ and let T be a linearly independent subset of $\mathcal{V}$. Then $L(S)$ spans $\mathcal{W}$ and $L(T)$ is linearly independent.

Definition Let $\mathcal{V}$ and $\mathcal{W}$ be vector spaces. Then $\mathcal{V}$ is **isomorphic** to $\mathcal{W}$, denoted $\mathcal{V} \cong \mathcal{W}$, if and only if there exists an isomorphism $L\colon \mathcal{V} \to \mathcal{W}$.

Theorem 5.17 Suppose $\mathcal{V} \cong \mathcal{W}$ and $\mathcal{V}$ is finite dimensional. Then $\mathcal{W}$ is finite dimensional and $\dim(\mathcal{V}) = \dim(\mathcal{W})$.

Theorem 5.18 If $\mathcal{V}$ is any n-dimensional vector space, then $\mathcal{V} \cong \mathbb{R}^n$.

Proof. Suppose that $\mathcal{V}$ is a vector space with $\dim(\mathcal{V}) = n$. If we can find an isomorphism $L: \mathcal{V} \to \mathbb{R}^n$, then $\mathcal{V} \cong \mathbb{R}^n$, and we will be done. Let $B = (\mathbf{v}_1, \ldots, \mathbf{v}_n)$ be an ordered basis for $\mathcal{V}$. Consider the mapping $L(\mathbf{v}) = [\mathbf{v}]_B$, for all $\mathbf{v} \in \mathcal{V}$. Now, L is a linear transformation by Example 4 in Section 5.1. Also,

$$\mathbf{v} \in \ker(L) \;\Leftrightarrow\; [\mathbf{v}]_B = [0, \ldots, 0] \;\Leftrightarrow\; \mathbf{v} = 0\mathbf{v}_1 + \cdots + 0\mathbf{v}_n \;\Leftrightarrow\; \mathbf{v} = \mathbf{0}.$$

Hence, $\ker(L) = \{\mathbf{0}_{\mathcal{V}}\}$, and L is one-to-one.

If $\mathbf{a} = [a_1, \ldots, a_n] \in \mathbb{R}^n$, then $L(a_1\mathbf{v}_1 + \cdots + a_n\mathbf{v}_n) = [a_1, \ldots, a_n]$, showing that $\mathbf{a} \in \mathbf{range}(L)$. Hence, L is onto, and so L is an isomorphism. □

Corollary 5.19 Any two n-dimensional vector spaces $\mathcal{V}$ and $\mathcal{W}$ are isomorphic. That is, if $\dim(\mathcal{V}) = \dim(\mathcal{W})$, then $\mathcal{V} \cong \mathcal{W}$.

Example 4

Consider the subset $S = \{x^3 - 2x^2 + x - 2,\ x^3 + x^2 + x + 1,\ x^3 - 5x^2 + x - 5,\ x^3 - x^2 - x + 1\}$ of $\mathcal{P}_3$. We use the coordinatization isomorphism $L: \mathcal{P}_3 \to \mathbb{R}^4$ with respect to the standard basis of $\mathcal{P}_3$ to obtain $L(S) = \{[1, -2, 1, -2], [1, 1, 1, 1], [1, -5, 1, -5], [1, -1, -1, 1]\}$, a subset of $\mathbb{R}^4$ corresponding to S. Row reducing

$$\begin{bmatrix} 1 & -2 & 1 & -2 \\ 1 & 1 & 1 & 1 \\ 1 & -5 & 1 & -5 \\ 1 & -1 & -1 & 1 \end{bmatrix} \quad \text{to obtain} \quad \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

shows, by the Simplified Span Method, that $\mathbf{span}(\{[1,-2,1,-2],[1,1,1,1],[1,-5,1,-5],[1,-1,-1,1]\}) = \mathbf{span}(\{[1,0,0,1],\ [0,1,0,1],\ [0,0,1,-1]\})$. Since L^{-1} is an isomorphism, Theorem 5.16 shows that $L^{-1}(\{[1,0,0,1],[0,1,0,1],[0,0,1,-1]\}) = \{x^3 + 1,\ x^2 + 1,\ x - 1\}$ spans the same subspace of $\mathcal{P}_3$ that S does. That is, $\mathbf{span}(\{x^3 + 1,\ x^2 + 1,\ x - 1\}) = \mathbf{span}(S)$.

Similarly, row reducing

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ -2 & 1 & -5 & -1 \\ 1 & 1 & 1 & -1 \\ -2 & 1 & -5 & 1 \end{bmatrix} \quad \text{to obtain} \quad \begin{bmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

shows, by the Independence Test Method, that $\{[1,-2,1,-2],[1,1,1,1],[1,-1,-1,1]\}$ is a linearly independent subset of $\mathbb{R}^4$, and that $[1,-5,1,-5] = 2[1,-2,1,-2] - [1,1,1,1] + 0[1,-1,-1,1]$. Since L^{-1} is an isomorphism, Theorem 5.16 shows us that $L^{-1}\big(\{[1,-2,1,-2],[1,1,1,1],[1,-1,-1,1]\}\big) = \big\{x^3 - 2x^2 + x - 2, x^3 + x^2 + x + 1, x^3 - x^2 - x + 1\big\}$ is a linearly independent subset of $\mathcal{P}_3$. The fact that L^{-1} is a linear transformation also assures us that $x^3 - 5x^2 + x - 5 = 2\big(x^3 - 2x^2 + x - 2\big) - \big(x^3 + x^2 + x + 1\big) + 0\big(x^3 - x^2 - x + 1\big)$. ■

Theorem 5.20 Let $L: \mathcal{V} \to \mathcal{W}$ be a linear transformation between nontrivial finite dimensional vector spaces, and let $L_1: \mathcal{V} \to \mathbb{R}^n$ and $L_2: \mathcal{W} \to \mathbb{R}^m$ be coordinatization isomorphisms with respect to some ordered bases B and C for $\mathcal{V}$ and $\mathcal{W}$, respectively. Let $M = L_2 \circ L \circ L_1^{-1}: \mathbb{R}^n \to \mathbb{R}^m$, so that $M([\mathbf{v}]_B) = [L(\mathbf{v})]_C$. Then,

(1) $L_1^{-1}(\ker(M)) = \ker(L) \subseteq \mathcal{V}$,

(2) $L_2^{-1}(\text{range}(M)) = \text{range}(L) \subseteq \mathcal{W}$,

(3) $\dim(\ker(M)) = \dim(\ker(L))$, and

(4) $\dim(\text{range}(M)) = \dim(\text{range}(L))$.

Corollary 5.21 Let $\mathcal{V}$ and $\mathcal{W}$ be finite dimensional vector spaces with $\dim(\mathcal{V}) = \dim(\mathcal{W})$. Let $L: \mathcal{V} \to \mathcal{W}$ be a linear transformation. Then L is one-to-one if and only if L is onto.