One-to-One and Onto Linear Transformations

DEFINITION: Let $L : V \to W$ be a linear transformation. L is **onto** if and only if the range of L is all of the codomain W. That is, L is onto if each $\mathbf{w}$ in W is the image of *at least one* $\mathbf{v}$ in V.

Equivalently, L maps V onto W if and only if, for each $\mathbf{w}$ in the codomain W, there exists at least one solution of $L(\mathbf{v}) = \mathbf{w}$. "Does L map V onto W?" is an existence question. The mapping L is *not* onto when there is some $\mathbf{w}$ in W for which the equation $L(\mathbf{v}) = \mathbf{w}$ has no solution.

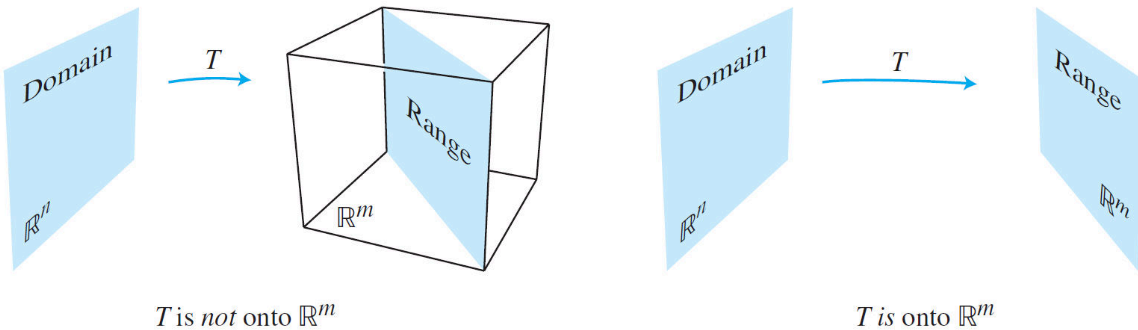

FIGURE 3 Is the range of T all of $\mathbb{R}^m$?

DEFINITION: Let $L : V \to W$ be a linear transformation. L is **one-to-one** if and only if distinct vectors in V have different images in W. That is, L is one-to-one if and only if each $\mathbf{w}$ in W is the image of *at most one* $\mathbf{v}$ in V.

Equivalently,

1. L is one-to-one if and only if, for all $\mathbf{v}_1, \mathbf{v}_2 \in V$, $L(\mathbf{v}_1) = L(\mathbf{v}_2)$ implies $\mathbf{v}_1 = \mathbf{v}_2$.

2. L is one-to-one if and only if, for each $\mathbf{w}$ in the codomain W, the equation $L(\mathbf{v}) = \mathbf{w}$ has either a unique solution or none at all.

"Is L one-to-one?" is an uniqueness question. The mapping L is *not* one-to-one when some $\mathbf{w}$ in W is the image of more than one vector in V. If there is no such $\mathbf{w}$, then L is one-to-one.

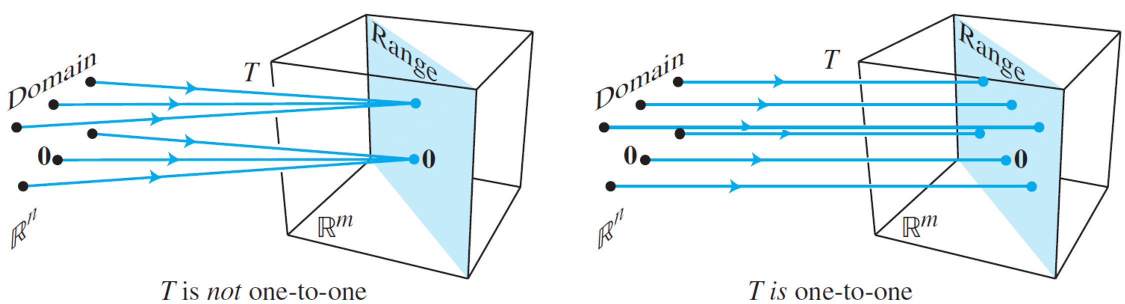

FIGURE 4 Is every **b** the image of at most one vector?

Theorem 5.12 Let $\mathcal{V}$ and $\mathcal{W}$ be vector spaces, and let $L: \mathcal{V} \to \mathcal{W}$ be a linear transformation. Then:

(1) L is one-to-one if and only if $\ker(L) = \{\mathbf{0}_{\mathcal{V}}\}$ (or, equivalently, if and only if $\dim(\ker(L)) = 0$), and

(2) If $\mathcal{W}$ is finite dimensional, then L is onto if and only if $\dim(\mathbf{range}(L)) = \dim(\mathcal{W})$.

Proof. First suppose that L is one-to-one, and let $\mathbf{v} \in \ker(L)$. We must show that $\mathbf{v} = \mathbf{0}_{\mathcal{V}}$. Now, $L(\mathbf{v}) = \mathbf{0}_{\mathcal{W}}$. However, by Theorem 5.1, $L(\mathbf{0}_{\mathcal{V}}) = \mathbf{0}_{\mathcal{W}}$. Because $L(\mathbf{v}) = L(\mathbf{0}_{\mathcal{V}})$ and L is one-to-one, we must have $\mathbf{v} = \mathbf{0}_{\mathcal{V}}$.

Conversely, suppose that $\ker(L) = \{\mathbf{0}_{\mathcal{V}}\}$. We must show that L is one-to-one. Let $\mathbf{v}_1, \mathbf{v}_2 \in \mathcal{V}$, with $L(\mathbf{v}_1) = L(\mathbf{v}_2)$. We must show that $\mathbf{v}_1 = \mathbf{v}_2$. Now, $L(\mathbf{v}_1) - L(\mathbf{v}_2) = \mathbf{0}_{\mathcal{W}}$, implying that $L(\mathbf{v}_1 - \mathbf{v}_2) = \mathbf{0}_{\mathcal{W}}$. Hence, $\mathbf{v}_1 - \mathbf{v}_2 \in \ker(L)$, by definition of the kernel. Since $\ker(L) = \{\mathbf{0}_{\mathcal{V}}\}$, $\mathbf{v}_1 - \mathbf{v}_2 = \mathbf{0}_{\mathcal{V}}$ and so $\mathbf{v}_1 = \mathbf{v}_2$.

Finally, note that, by definition, L is onto if and only if $\mathbf{range}(L) = \mathcal{W}$, and therefore part (2) of the theorem follows immediately from Theorem 4.16. □

EXAMPLE: Let L be a linear transformation whose matrix is

$$A = \begin{bmatrix} 1 & -4 & 8 & 1 \\ 0 & 2 & -1 & 3 \\ 0 & 0 & 0 & 5 \end{bmatrix}$$

Does L map $\mathbb{R}^4$ onto $\mathbb{R}^3$? Is L a one-to-one mapping?

Solution: Since A has a pivot in every row, T is onto. Since A does not have a pivot in every column, T is not one-to-one.

Example

Consider the linear transformation L: $\mathcal{M}_{22} \to \mathcal{M}_{23}$ given by $L\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = \begin{bmatrix} a-b & 0 & c-d \\ c+d & a+b & 0 \end{bmatrix}$. If $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \ker(L)$, then $a-b = c-d = c+d = a+b = 0$. Solving these equations yields $a = b = c = d = 0$, and so $\ker(L)$ contains only the zero matrix $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$; that is, $\dim(\ker(L)) = 0$. Thus, by part (1) of Theorem 5.12, L is one-to-one. However, by the Dimension Theorem, $\dim(\mathbf{range}(L)) = \dim(\mathcal{M}_{22}) - \dim(\ker(L)) = \dim(\mathcal{M}_{22}) = 4$. Hence, by part (2) of Theorem 5.12, L is not onto. In particular, $\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \notin \mathbf{range}(L)$.

On the other hand, consider M: $\mathcal{M}_{23} \to \mathcal{M}_{22}$ given by $M\left(\begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix}\right) = \begin{bmatrix} a+b & a+c \\ d+e & d+f \end{bmatrix}$. It is easy to see that M is onto, since $M\left(\begin{bmatrix} 0 & b & c \\ 0 & e & f \end{bmatrix}\right) = \begin{bmatrix} b & c \\ e & f \end{bmatrix}$, and thus every 2×2 matrix is in $\mathbf{range}(M)$. Thus, by part (2) of Theorem 5.12, $\dim(\mathbf{range}(M)) = \dim(\mathcal{M}_{22}) = 4$. Then, by the Dimension Theorem, $\ker(M) = \dim(\mathcal{M}_{23}) - \dim(\mathbf{range}(M)) = 6 - 4 = 2$. Hence, by part (1) of Theorem 5.12, M is not one-to-one. In particular, $\begin{bmatrix} 1 & -1 & -1 \\ 1 & -1 & -1 \end{bmatrix} \in \ker(L)$.

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Theorem 5.13 Let $\mathcal{V}$ and $\mathcal{W}$ be vector spaces, and let $L: \mathcal{V} \to \mathcal{W}$ be a linear transformation. Then:

(1) If L is one-to-one, and T is a linearly independent subset of $\mathcal{V}$, then $L(T)$ is linearly independent in $\mathcal{W}$.

(2) If L is onto, and S spans $\mathcal{V}$, then $L(S)$ spans $\mathcal{W}$.

Proof. Suppose that L is one-to-one, and T is a linearly independent subset of $\mathcal{V}$. To prove that $L(T)$ is linearly independent in $\mathcal{W}$, it is enough to show that any finite subset of $L(T)$ is linearly independent. Suppose $\{L(\mathbf{x}_1), \ldots, L(\mathbf{x}_n)\}$ is a finite subset of $L(T)$, for vectors $\mathbf{x}_1, \ldots, \mathbf{x}_n \in T$, and suppose $b_1 L(\mathbf{x}_1) + \cdots + b_n L(\mathbf{x}_n) = \mathbf{0}_{\mathcal{W}}$. Then, $L(b_1\mathbf{x}_1 + \cdots + b_n\mathbf{x}_n) = \mathbf{0}_{\mathcal{W}}$, implying that $b_1\mathbf{x}_1 + \cdots + b_n\mathbf{x}_n \in \ker(L)$. But since L is one-to-one, Theorem 5.12 tells us that $\ker(L) = \{\mathbf{0}_{\mathcal{V}}\}$. Hence, $b_1\mathbf{x}_1 + \cdots + b_n\mathbf{x}_n = \mathbf{0}_{\mathcal{V}}$. Then, because the vectors in T are linearly independent, $b_1 = b_2 = \cdots = b_n = 0$. Therefore, $\{L(\mathbf{x}_1), \ldots, L(\mathbf{x}_n)\}$ is linearly independent. Hence, $L(T)$ is linearly independent.

Now suppose that L is onto, and S spans $\mathcal{V}$. To prove that $L(S)$ spans $\mathcal{W}$, we must show that any vector $\mathbf{w} \in \mathcal{W}$ can be expressed as a linear combination of vectors in $L(S)$. Since L is onto, there is a $\mathbf{v} \in \mathcal{V}$ such that $L(\mathbf{v}) = \mathbf{w}$. Since S spans $\mathcal{V}$, there are scalars $a_1, \ldots, a_n$ and vectors $\mathbf{v}_1, \ldots, \mathbf{v}_n \in S$ such that $\mathbf{v} = a_1\mathbf{v}_1 + \cdots + a_n\mathbf{v}_n$. Thus, $\mathbf{w} = L(\mathbf{v}) = L(a_1\mathbf{v}_1 + \cdots + a_n\mathbf{v}_n) = a_1L(\mathbf{v}_1) + \cdots + a_nL(\mathbf{v}_n)$. Hence, $L(S)$ spans $\mathcal{W}$. $\square$