The Dimension Theorem

Definition Let $L: \mathcal{V} \to \mathcal{W}$ be a linear transformation. The **kernel** of L, denoted by $\ker(L)$, is the subset of all vectors in $\mathcal{V}$ that map to $\mathbf{0}_{\mathcal{W}}$. That is, $\ker(L) = \{\mathbf{v} \in \mathcal{V} \mid L(\mathbf{v}) = \mathbf{0}_{\mathcal{W}}\}$. The **range** of L, or, range(L), is the subset of all vectors in $\mathcal{W}$ that are the image of some vector in $\mathcal{V}$. That is, range$(L) = \{L(\mathbf{v}) \mid \mathbf{v} \in \mathcal{V}\}$.

Theorem 5.8 If $L: \mathcal{V} \to \mathcal{W}$ is a linear transformation, then the kernel of L is a subspace of $\mathcal{V}$ and the range of L is a subspace of $\mathcal{W}$.

Example 1

Projection: For $n \geq 3$, consider the linear operator $L: \mathbb{R}^n \to \mathbb{R}^n$ given by $L([a_1, a_2, \ldots, a_n]) = [a_1, a_2, 0, \ldots, 0]$. Now, $\ker(L)$ consists of those elements of the domain that map to $[0, 0, \ldots, 0]$, the zero vector of the codomain. Hence, for vectors in the kernel, $a_1 = a_2 = 0$, but $a_3, \ldots, a_n$ can have any values. Thus,

$$\ker(L) = \left\{ [0, 0, a_3, \ldots, a_n] \middle| a_3, \ldots, a_n \in \mathbb{R} \right\}.$$

Notice that $\ker(L)$ is a subspace of the domain and that $\dim(\ker(L)) = n - 2$, because the standard basis vectors $\mathbf{e}_3, \ldots, \mathbf{e}_n$ of $\mathbb{R}^n$ span $\ker(L)$.

Also, range(L) consists of those elements of the codomain $\mathbb{P}^2$ that are images of domain elements. Hence, range$(L) = \{[a_1, a_2, 0, \ldots, 0] | a_1, a_2 \in \mathbb{R}\}$. Notice that range$(L)$ is a subspace of the codomain and that $\dim(\text{range}(L)) = 2$, since the standard basis vectors $\mathbf{e}_1$ and $\mathbf{e}_2$ span range(L).

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Example 2

Differentiation: Consider the linear transformation $L: \mathcal{P}_3 \to \mathcal{P}_2$ given by $L(ax^3 + bx^2 + cx + d) = 3ax^2 + 2bx + c$. Now, $\ker(L)$ consists of the polynomials in $\mathcal{P}_3$ that map to the zero polynomial in $\mathcal{P}_2$. However, if $3ax^2 + 2bx + c = 0$, we must have $a = b = c = 0$. Hence, $\ker(L) = \left\{ 0x^3 + 0x^2 + 0x + d \middle| d \in \mathbb{R} \right\}$; that is, $\ker(L)$ is just the subset of $\mathcal{P}_3$ of all constant polynomials. Notice that $\ker(L)$ is a subspace of $\mathcal{P}_3$ and that $\dim(\ker(L)) = 1$ because the single polynomial "1" spans $\ker(L)$.

Finding the Kernel and the Range from the Matrix of a Linear Transformation

Method for Finding a Basis for the Kernel of a Linear Transformation (Kernel Method)
Let $L: \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation given by $\mathbf{L(X)} = \mathbf{AX}$ for some $m \times n$ matrix $\mathbf{A}$. To find a basis for $\ker(L)$, perform the following steps:

Step 1: Find $\mathbf{B}$, the reduced row echelon form of $\mathbf{A}$.

Step 2: Solve for one particular solution for each independent variable in the homogeneous system $\mathbf{BX} = \mathbf{O}$. The ith such solution, $\mathbf{v}_i$, is found by setting the ith independent variable equal to 1 and setting all other independent variables equal to 0.

Step 3: The set $\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ is a basis for $\ker(L)$. (We can replace any $\mathbf{v}_i$ with $c\mathbf{v}_i$, where $c \neq 0$, to eliminate fractions.)

Method for Finding a Basis for the Range of a Linear Transformation (Range Method)
Let $L: \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation given by $\mathbf{L(X)} = \mathbf{AX}$, for some $m \times n$ matrix $\mathbf{A}$. To find a basis for $\mathrm{range}(L)$, perform the following steps:

Step 1: Find $\mathbf{B}$, the reduced row echelon form of $\mathbf{A}$.

Step 2: Form the set of those columns of $\mathbf{A}$ whose corresponding columns in $\mathbf{B}$ have nonzero pivots. This set is a basis for $\mathrm{range}(L)$.

EXAMPLE: Let $L : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation given by $L(\mathbf{x}) = A\mathbf{x}$, where

$$A = \begin{bmatrix} -3 & 6 & -1 & 1 & -7 \\ 1 & -2 & 2 & 3 & -1 \\ 2 & -4 & 5 & 8 & -4 \end{bmatrix}$$

Find bases for $\ker(L)$ and $\mathrm{range}(L)$.

Solution: We use row operations

$$\begin{bmatrix} -3 & 6 & -1 & 1 & -7 & 0 \\ 1 & -2 & 2 & 3 & -1 & 0 \\ 2 & -4 & 5 & 8 & -4 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 & 2 & 3 & -1 & 0 \\ -3 & 6 & -1 & 1 & -7 & 0 \\ 2 & -4 & 5 & 8 & -4 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 & 2 & 3 & -1 & 0 \\ 0 & 0 & 5 & 10 & -10 & 0 \\ 0 & 0 & 1 & 2 & -2 & 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & -2 & 2 & 3 & -1 & 0 \\ 0 & 0 & 1 & 2 & -2 & 0 \\ 0 & 0 & 1 & 2 & -2 & 0 \end{bmatrix}$$

$$\sim \underbrace{\begin{bmatrix} 1 & -2 & 2 & 3 & -1 & 0 \\ 0 & 0 & 1 & 2 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}}_{\text{Echelon Form}}$$

Therefore

$$\left\{ \begin{bmatrix} -3 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} -1 \\ 2 \\ 5 \end{bmatrix} \right\}$$

a basis for range(L). To find a basis for $\ker(L)$ we note that

$$\underbrace{\begin{bmatrix} 1 & -2 & 2 & 3 & -1 & 0 \\ 0 & 0 & 1 & 2 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}}_{\text{Echelon Form}} \sim \underbrace{\begin{bmatrix} 1 & -2 & 0 & -1 & 3 & 0 \\ 0 & 0 & 1 & 2 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}}_{\text{Reduced Echelon Form}}$$

From this it follows that the system $A\mathbf{x} = \mathbf{0}$ has the same solution set as the system

$$\begin{cases} x_1 - 2x_2 - x_4 + 3x_5 = 0 \\ x_3 + 2x_4 - 2x_5 = 0 \end{cases} \implies \begin{cases} x_1 = 2x_2 + x_4 - 3x_5 \\ x_3 = -2x_4 + 2x_5 \end{cases}$$

hence

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 2x_2 + x_4 - 3x_5 \\ x_2 \\ -2x_4 + 2x_5 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 2 \cdot x_2 + 1 \cdot x_4 + (-3) \cdot x_5 \\ 1 \cdot x_2 + 0 \cdot x_4 + 0 \cdot x_5 \\ 0 \cdot x_2 + (-2) \cdot x_4 + 2 \cdot x_5 \\ 0 \cdot x_2 + 1 \cdot x_4 + 0 \cdot x_5 \\ 0 \cdot x_2 + 0 \cdot x_4 + 1 \cdot x_5 \end{bmatrix}$$

$$= x_2 \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 1 \\ 0 \\ -2 \\ 1 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} -3 \\ 0 \\ 2 \\ 0 \\ 1 \end{bmatrix}$$

Therefore

$$\left\{ \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ 2 \\ 0 \\ 1 \end{bmatrix} \right\}$$

a basis for $\ker(L)$.

EXAMPLE: Let $L : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation given by $L(\mathbf{x}) = A\mathbf{x}$, where

$$A = \begin{bmatrix} 2 & 4 & -2 & 1 \\ -2 & -5 & 7 & 3 \\ 3 & 7 & -8 & 6 \end{bmatrix}$$

Find bases for $\ker(L)$ and range(L).

Solution: We use row operations

$$\begin{bmatrix} 2 & 4 & -2 & 1 & 0 \\ -2 & -5 & 7 & 3 & 0 \\ 3 & 7 & -8 & 6 & 0 \end{bmatrix} \sim \begin{bmatrix} 2 & 4 & -2 & 1 & 0 \\ 0 & -1 & 5 & 4 & 0 \\ 1 & 3 & -6 & 5 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 & -6 & 5 & 0 \\ 2 & 4 & -2 & 1 & 0 \\ 0 & -1 & 5 & 4 & 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 3 & -6 & 5 & 0 \\ 0 & -2 & 10 & -9 & 0 \\ 0 & -1 & 5 & 4 & 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 3 & -6 & 5 & 0 \\ 0 & 0 & 0 & -17 & 0 \\ 0 & -1 & 5 & 4 & 0 \end{bmatrix}$$

$$\sim \underbrace{\begin{bmatrix} 1 & 3 & -6 & 5 & 0 \\ 0 & -1 & 5 & 4 & 0 \\ 0 & 0 & 0 & -17 & 0 \end{bmatrix}}_{\text{Echelon Form}}$$

Therefore

$$\left\{ \begin{bmatrix} 2 \\ -2 \\ 3 \end{bmatrix}, \begin{bmatrix} 4 \\ -5 \\ 7 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \\ 6 \end{bmatrix} \right\}$$

a basis for range(L). To find a basis for ker(L) we note that

$$\underbrace{\begin{bmatrix} 1 & 3 & -6 & 5 & 0 \\ 0 & -1 & 5 & 4 & 0 \\ 0 & 0 & 0 & -17 & 0 \end{bmatrix}}_{\text{Echelon Form}} \sim \begin{bmatrix} 1 & 3 & -6 & 5 & 0 \\ 0 & 1 & -5 & -4 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 & -6 & 0 & 0 \\ 0 & 1 & -5 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix} \sim \underbrace{\begin{bmatrix} 1 & 0 & 9 & 0 & 0 \\ 0 & 1 & -5 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}}_{\text{Reduced Echelon Form}}$$

From this it follows that the system $A\mathbf{x} = \mathbf{0}$ has the same solution set as the system

$$\begin{cases} x_1 + 9x_3 = 0 \\ x_2 - 5x_3 = 0 \\ x_4 = 0 \end{cases} \implies \begin{cases} x_1 = -9x_3 \\ x_2 = 5x_3 \\ x_4 = 0 \end{cases}$$

hence

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -9x_3 \\ 5x_3 \\ x_3 \\ 0 \end{bmatrix} = x_3 \begin{bmatrix} -9 \\ 5 \\ 1 \\ 0 \end{bmatrix}$$

Therefore

$$\left\{ \begin{bmatrix} -9 \\ 5 \\ 1 \\ 0 \end{bmatrix} \right\}$$

a basis for ker(L).

EXAMPLE: Let $L : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation given by $L(\mathbf{x}) = A\mathbf{x}$, where

$$A = \begin{bmatrix} 1 & 4 & 0 & 2 & -1 \\ 3 & 12 & 1 & 5 & 5 \\ 2 & 8 & 1 & 3 & 2 \\ 5 & 20 & 2 & 8 & 8 \end{bmatrix}$$

It can be shown that A is row equivalent to the matrix reduced echelon form

$$\begin{bmatrix} 1 & 4 & 0 & 2 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Find bases for $\ker(L)$ and $\text{range}(L)$.

Solution: From the reduced echelon form it immediately follows that

$$\left\{ \begin{bmatrix} 1 \\ 3 \\ 2 \\ 5 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} -1 \\ 5 \\ 2 \\ 8 \end{bmatrix} \right\}$$

a basis for $\text{range}(L)$. To find a basis for $\ker(L)$ we note that the system $A\mathbf{x} = \mathbf{0}$ has the same solution set as the system

$$\begin{cases} x_1 + 4x_2 + 2x_4 = 0 \\ x_3 - x_4 = 0 \\ x_5 = 0 \end{cases} \Longrightarrow \begin{cases} x_1 = -4x_2 - 2x_4 \\ x_3 = x_4 \\ x_5 = 0 \end{cases}$$

hence

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} -4x_2 - 2x_4 \\ x_2 \\ x_4 \\ x_4 \\ 0 \end{bmatrix} = x_2 \begin{bmatrix} -4 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -2 \\ 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}$$

Therefore

$$\left\{ \begin{bmatrix} -4 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ 1 \\ 1 \\ 0 \end{bmatrix} \right\}$$

a basis for $\ker(L)$.

EXAMPLE: Let

$$\mathbf{v}_1=\begin{bmatrix}-1\\2\\0\end{bmatrix},\quad \mathbf{v}_2=\begin{bmatrix}4\\1\\9\end{bmatrix},\quad \mathbf{v}_3=\begin{bmatrix}-2\\1\\-3\end{bmatrix},\quad \mathbf{v}_4=\begin{bmatrix}0\\1\\-1\end{bmatrix},\quad \mathbf{v}_5=\begin{bmatrix}-3\\0\\-6\end{bmatrix}$$

and let $L:\mathbb{R}^n\to\mathbb{R}^m$ be a linear transformation given by $L(\mathbf{x})=A\mathbf{x}$, where

$$A=[\mathbf{v}_1\ \ \mathbf{v}_2\ \ \mathbf{v}_3\ \ \mathbf{v}_4\ \ \mathbf{v}_5]$$

Find bases for $\ker(L)$ and $\operatorname{range}(L)$.

Solution: We use row operations

$$\begin{bmatrix}-1&4&-2&0&-3&0\\2&1&1&-1&0&0\\0&9&-3&-1&-6&0\end{bmatrix}\sim\begin{bmatrix}-1&4&-2&0&-3&0\\0&9&-3&-1&-6&0\\0&9&-3&-1&-6&0\end{bmatrix}\sim\underbrace{\begin{bmatrix}1&-4&2&0&3&0\\0&9&-3&-1&-6&0\\0&0&0&0&0&0\end{bmatrix}}_{\text{Echelon Form}}$$

Therefore $\{\mathbf{v}_1,\mathbf{v}_2\}$ is a basis for $\operatorname{range}(L)$. To find a basis for $\ker(L)$ we note that

$$\underbrace{\begin{bmatrix}1&-4&2&0&3&0\\0&9&-3&-1&-6&0\\0&0&0&0&0&0\end{bmatrix}}_{\text{Echelon Form}}\sim\underbrace{\begin{bmatrix}1&0&\frac{2}{3}&-\frac{4}{9}&\frac{1}{3}&0\\0&1&-\frac{1}{3}&-\frac{1}{9}&-\frac{2}{3}&0\\0&0&0&0&0&0\end{bmatrix}}_{\text{Reduced Echelon Form}}$$

From this it follows that the system $A\mathbf{x}=\mathbf{0}$ has the same solution set as the system

$$\begin{cases}x_1+\frac{2}{3}x_3-\frac{4}{9}x_4+\frac{1}{3}x_5=0\\[2mm] x_2-\frac{1}{3}x_3-\frac{1}{9}x_4-\frac{2}{3}x_5=0\end{cases}\qquad\Longrightarrow\qquad\begin{cases}x_1=-\frac{2}{3}x_3+\frac{4}{9}x_4-\frac{1}{3}x_5\\[2mm] x_2=\frac{1}{3}x_3+\frac{1}{9}x_4+\frac{2}{3}x_5\end{cases}$$

hence

$$\begin{bmatrix}x_1\\x_2\\x_3\\x_4\\x_5\end{bmatrix}=\begin{bmatrix}-\frac{2}{3}x_3+\frac{4}{9}x_4-\frac{1}{3}x_5\\ \frac{1}{3}x_3+\frac{1}{9}x_4+\frac{2}{3}x_5\\x_3\\x_4\\x_5\end{bmatrix}=x_3\underbrace{\begin{bmatrix}-\frac{2}{3}\\ \frac{1}{3}\\1\\0\\0\end{bmatrix}}_{\mathbf{u}_1}+x_4\underbrace{\begin{bmatrix}\frac{4}{9}\\ \frac{1}{9}\\0\\1\\0\end{bmatrix}}_{\mathbf{u}_2}+x_5\underbrace{\begin{bmatrix}-\frac{1}{3}\\ \frac{2}{3}\\0\\0\\1\end{bmatrix}}_{\mathbf{u}_3}$$

Therefore

$$\{\mathbf{u}_1,\mathbf{u}_2,\mathbf{u}_3\}$$

a basis for $\ker(L)$.

Theorem 5.9 If $L: \mathbb{R}^n \to \mathbb{R}^m$ is a linear transformation with matrix $\mathbf{A}$ with respect to any bases for $\mathbb{R}^n$ and $\mathbb{R}^m$, then

(1) $\dim(\mathrm{range}(L)) = \mathrm{rank}\,(\mathbf{A})$
(2) $\dim(\ker(L)) = n - \mathrm{rank}\,(\mathbf{A})$
(3) $\dim(\ker(L)) + \dim(\mathrm{range}(L)) = \dim(\mathrm{domain}(L)) = n$.

Theorem 5.10 (Dimension Theorem) If $L: \mathcal{V} \to \mathcal{W}$ is a linear transformation and $\mathcal{V}$ is finite dimensional, then $\mathrm{range}(L)$ is finite dimensional, and

$$\dim(\ker(L)) + \dim(\mathrm{range}(L)) = \dim(\mathcal{V}).$$

Example 6

Consider $L: \mathcal{M}_{nn} \to \mathcal{M}_{nn}$ given by $L(\mathbf{A}) = \mathbf{A} + \mathbf{A}^T$. Now, $\ker(L) = \{\mathbf{A} \in \mathcal{M}_{nn} \,|\, \mathbf{A} + \mathbf{A}^T = \mathbf{O}_n\}$. However, $\mathbf{A} + \mathbf{A}^T = \mathbf{O}_n$ implies that $\mathbf{A} = -\mathbf{A}^T$. Hence, $\ker(L)$ is precisely the set of all skew-symmetric $n \times n$ matrices.

The range of L is the set of all matrices $\mathbf{B}$ of the form $\mathbf{A} + \mathbf{A}^T$ for some $n \times n$ matrix $\mathbf{A}$. However, if $\mathbf{B} = \mathbf{A} + \mathbf{A}^T$, then $\mathbf{B}^T = \left(\mathbf{A} + \mathbf{A}^T\right)^T = \mathbf{A}^T + \mathbf{A} = \mathbf{B}$, so $\mathbf{B}$ is symmetric. Thus, $\mathrm{range}(L) \subseteq$ {symmetric $n \times n$ matrices}.

Next, if $\mathbf{B}$ is a symmetric $n \times n$ matrix, then $L(\frac{1}{2}\mathbf{B}) = \frac{1}{2}L(\mathbf{B}) = \frac{1}{2}(\mathbf{B} + \mathbf{B}^T) = \frac{1}{2}(\mathbf{B} + \mathbf{B}) = \mathbf{B}$, and so $\mathbf{B} \in \mathrm{range}(L)$, thus proving {symmetric $n \times n$ matrices} $\subseteq \mathrm{range}(L)$. Hence, $\mathrm{range}(L)$ is the set of all symmetric $n \times n$ matrices.

In Exercise 12 of Section 4.6, we found that $\dim$({skew-symmetric $n \times n$ matrices}) $= (n^2 - n)/2$ and that $\dim$({symmetric $n \times n$ matrices}) $= \left(n^2 + n\right)/2$. Notice that the Dimension Theorem holds here, since $\dim(\ker(L)) + \dim(\mathrm{range}(L)) = (n^2 - n)/2 + \left(n^2 + n\right)/2 = n^2 = \dim\,(\mathcal{M}_{nn})$. ■

Corollary 5.11 If $\mathbf{A}$ is any matrix, then $\mathrm{rank}(\mathbf{A}) = \mathrm{rank}(\mathbf{A}^T)$.

Proof. Let $\mathbf{A}$ be an $m \times n$ matrix. Consider the linear transformation $L: \mathbb{R}^n \to \mathbb{R}^m$ with associated matrix $\mathbf{A}$ (using the standard bases). By the Range Method, $\mathrm{range}(L)$ is the span of the column vectors of $\mathbf{A}$. Hence, $\mathrm{range}(L)$ is the span of the row vectors of $\mathbf{A}^T$; that is, $\mathrm{range}(L)$ is the row space of $\mathbf{A}^T$. Thus, $\dim(\mathrm{range}(L)) = \mathrm{rank}(\mathbf{A}^T)$, by the Simplified Span Method. But by Theorem 5.9, $\dim(\mathrm{range}(L)) = \mathrm{rank}(\mathbf{A})$. Hence, $\mathrm{rank}(\mathbf{A}) = \mathrm{rank}(\mathbf{A}^T)$. □