The Matrix of a Linear Transformation

EXAMPLE 1 The columns of $I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ are $\mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\mathbf{e}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. Suppose T is a linear transformation from $\mathbb{R}^2$ into $\mathbb{R}^3$ such that

$$T(\mathbf{e}_1) = \begin{bmatrix} 5 \\ -7 \\ 2 \end{bmatrix} \quad \text{and} \quad T(\mathbf{e}_2) = \begin{bmatrix} -3 \\ 8 \\ 0 \end{bmatrix}$$

With no additional information, find a formula for the image of an arbitrary $\mathbf{x}$ in $\mathbb{R}^2$.

SOLUTION Write

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix} = x_1\mathbf{e}_1 + x_2\mathbf{e}_2 \tag{1}$$

Since T is a *linear* transformation,

$$\begin{aligned} T(\mathbf{x}) &= x_1 T(\mathbf{e}_1) + x_2 T(\mathbf{e}_2) \qquad (2) \\ &= x_1 \begin{bmatrix} 5 \\ -7 \\ 2 \end{bmatrix} + x_2 \begin{bmatrix} -3 \\ 8 \\ 0 \end{bmatrix} = \begin{bmatrix} 5x_1 - 3x_2 \\ -7x_1 + 8x_2 \\ 2x_1 + 0 \end{bmatrix} \end{aligned}$$

■

Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation. Then there exists a unique matrix A such that

$$T(\mathbf{x}) = A\mathbf{x} \quad \text{for all } \mathbf{x} \text{ in } \mathbb{R}^n$$

In fact, A is the $m \times n$ matrix whose jth column is the vector $T(\mathbf{e}_j)$, where $\mathbf{e}_j$ is the jth column of the identity matrix in $\mathbb{R}^n$:

$$A = \begin{bmatrix} T(\mathbf{e}_1) & \cdots & T(\mathbf{e}_n) \end{bmatrix} \tag{3}$$

PROOF Write $\mathbf{x} = I_n\mathbf{x} = [\,\mathbf{e}_1 \quad \cdots \quad \mathbf{e}_n\,]\mathbf{x} = x_1\mathbf{e}_1 + \cdots + x_n\mathbf{e}_n$, and use the linearity of T to compute

$$\begin{aligned} T(\mathbf{x}) &= T(x_1\mathbf{e}_1 + \cdots + x_n\mathbf{e}_n) = x_1T(\mathbf{e}_1) + \cdots + x_nT(\mathbf{e}_n) \\ &= \begin{bmatrix} T(\mathbf{e}_1) & \cdots & T(\mathbf{e}_n) \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = A\mathbf{x} \end{aligned}$$

The uniqueness of A is treated in Exercise 33. ■

The matrix A in (3) is called the **standard matrix for the linear transformation** T.

EXAMPLE 2 Find the standard matrix A for the dilation transformation $T(\mathbf{x}) = 3\mathbf{x}$, for $\mathbf{x}$ in $\mathbb{R}^2$.

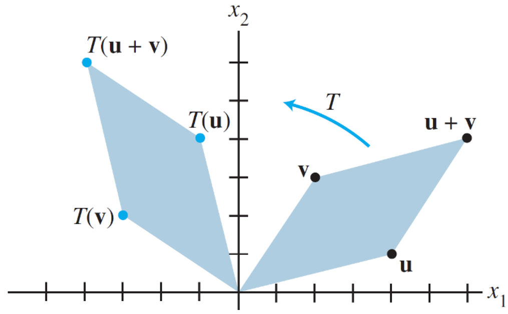
SOLUTION Write

$$T(\mathbf{e}_1) = 3\mathbf{e}_1 = \begin{bmatrix} 3 \\ 0 \end{bmatrix} \quad \text{and} \quad T(\mathbf{e}_2) = 3\mathbf{e}_2 = \begin{bmatrix} 0 \\ 3 \end{bmatrix}$$

$$A = \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}$$

■

EXAMPLE: Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the transformation that rotates each point in $\mathbb{R}^2$ about the origin through an angle ϕ, with counterclockwise rotation for a positive angle. We could show geometrically that such a transformation is linear.



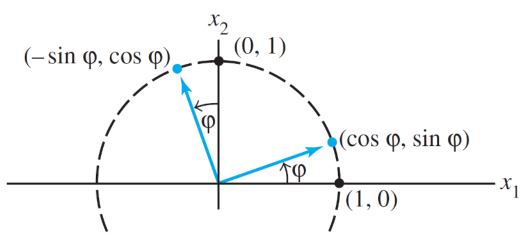
Find the standard matrix A of this transformation.

Solution: Note that

$$\mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \text{rotates into} \quad \begin{bmatrix} \cos\phi \\ \sin\phi \end{bmatrix}$$

and

$$\mathbf{e}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad \text{rotates into} \quad \begin{bmatrix} -\sin\phi \\ \cos\phi \end{bmatrix}$$



By the Theorem above,

$$A = \begin{bmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{bmatrix}$$

EXAMPLE: Let $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ and

$$A_1 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad A_2 = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$$

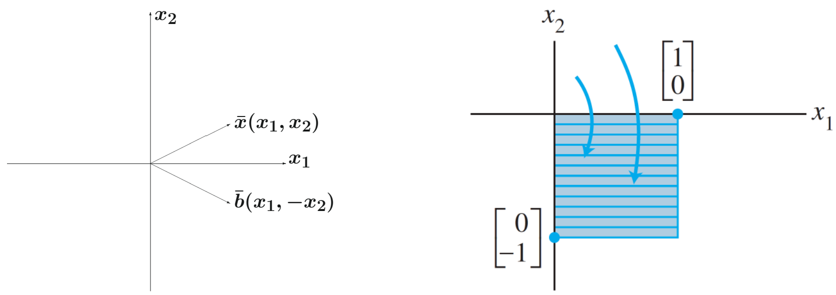
$$A_3 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad A_4 = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}, \quad A_5 = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$

$$B_1 = \begin{bmatrix} k & 0 \\ 0 & 1 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 1 & 0 \\ 0 & k \end{bmatrix}, \quad C_1 = \begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix}, \quad C_2 = \begin{bmatrix} 1 & 0 \\ k & 1 \end{bmatrix}$$

Find $A_i\mathbf{x}$, $B_i\mathbf{x}$, $C_i\mathbf{x}$. Provide illustrations and geometric explanations.

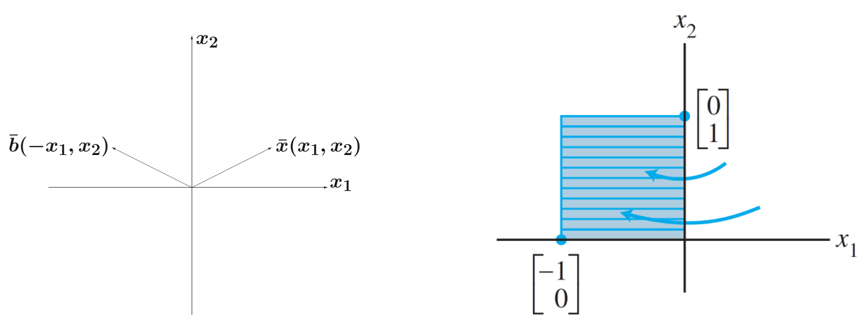
Solution:

1. $A_1\mathbf{x} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ -x_2 \end{bmatrix}$



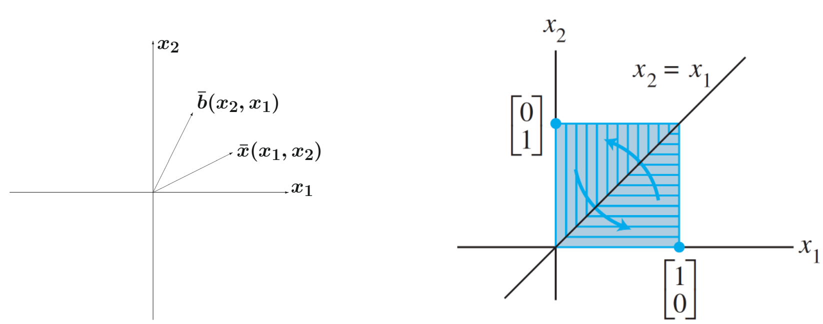
Reflection through the x_1-axis

2. $A_2\mathbf{x} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -x_1 \\ x_2 \end{bmatrix}$



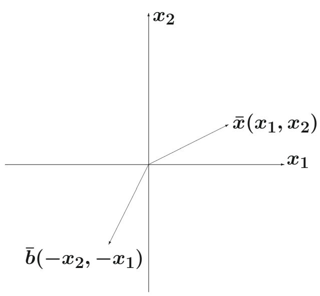
Reflection through the x_2-axis

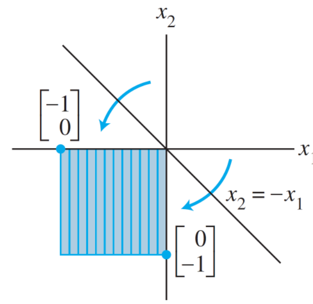
3. $A_3\mathbf{x} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_2 \\ x_1 \end{bmatrix}$



Reflection through the line $x_2 = x_1$

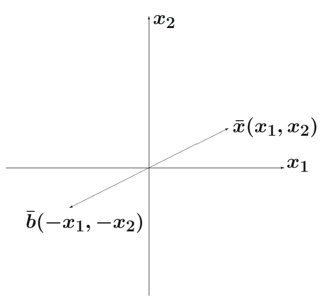
4. $A_4\mathbf{x} = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -x_2 \\ -x_1 \end{bmatrix}$

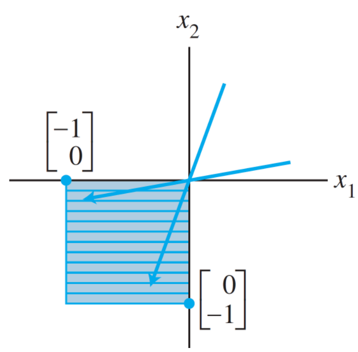




Reflection through the line $x_2 = -x_1$

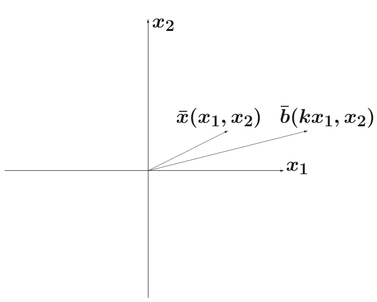
5. $A_5\mathbf{x} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -x_1 \\ -x_2 \end{bmatrix}$

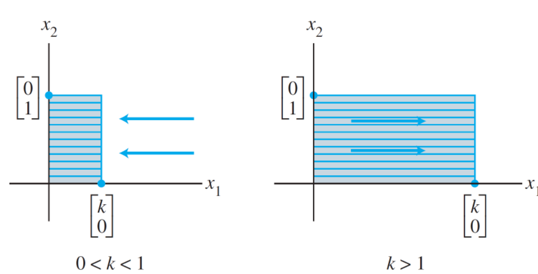




Reflection through the origin

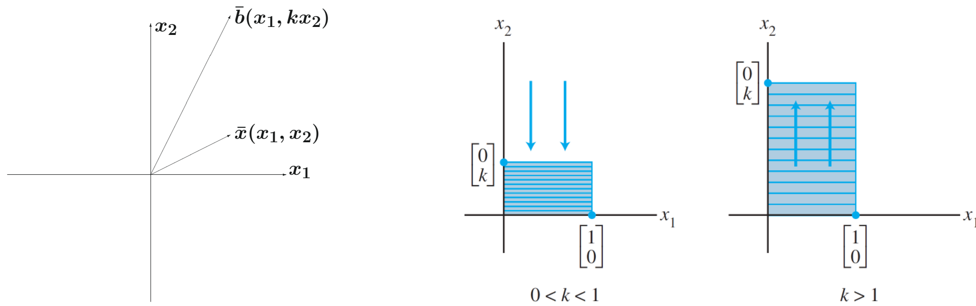
6. $B_1\mathbf{x} = \begin{bmatrix} k & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} kx_1 \\ x_2 \end{bmatrix}$





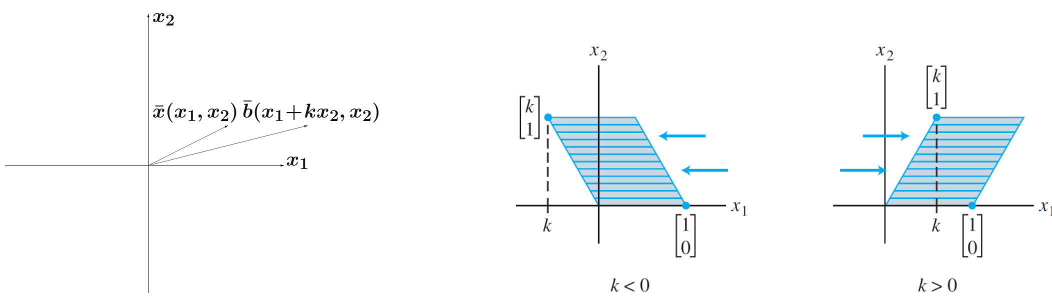
Horizontal expansion

7. $B_2\mathbf{x} = \begin{bmatrix} 1 & 0 \\ 0 & k \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ kx_2 \end{bmatrix}$



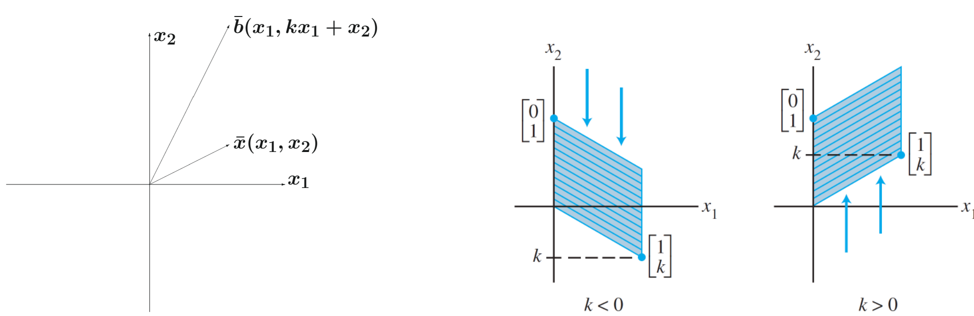
Vertical expansion

8. $C_1\mathbf{x} = \begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 + kx_2 \\ x_2 \end{bmatrix}$



Horizontal shear

9. $C_2\mathbf{x} = \begin{bmatrix} 1 & 0 \\ k & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ kx_1 + x_2 \end{bmatrix}$



Vertical shear

EXAMPLE: Let $\mathcal{E} = \{\mathbf{e}_1, \ \mathbf{e}_2\}$ be a basis for a vector space V. Let also $T : V \to V$ be a linear transformation such that

$$T(\mathbf{e}_1) = 3\mathbf{e}_1 - 2\mathbf{e}_2, \qquad T(\mathbf{e}_2) = 4\mathbf{e}_1 + 7\mathbf{e}_2$$

Find the matrix M for the linear transformation T relative to the basis $\mathcal{E}$.

Solution: Let $\mathbf{x} = x_1\mathbf{e}_1 + x_2\mathbf{e}_2$, then

$$\begin{aligned} T(\mathbf{x}) &= T(x_1\mathbf{e}_1 + x_2\mathbf{e}_2) \\ &= T(x_1\mathbf{e}_1) + T(x_2\mathbf{e}_2) \\ &= x_1T(\mathbf{e}_1) + x_2T(\mathbf{e}_2) \\ &= x_1(3\mathbf{e}_1 - 2\mathbf{e}_2) + x_2(4\mathbf{e}_1 + 7\mathbf{e}_2) \\ &= (3x_1 + 4x_2)\mathbf{e}_1 + (-2x_1 + 7x_2)\mathbf{e}_2 \end{aligned}$$

Therefore

$$T(\mathbf{x}) = \begin{bmatrix} 3x_1 + 4x_2 \\ -2x_1 + 7x_2 \end{bmatrix} = \begin{bmatrix} 3 & 4 \\ -2 & 7 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

So, $M = \begin{bmatrix} 3 & 4 \\ -2 & 7 \end{bmatrix}$.

EXAMPLE: Let $\mathcal{B} = \{\mathbf{b}_1, \ \mathbf{b}_2\}$ be a basis for V and $\mathcal{C} = \{\mathbf{c}_1, \ \mathbf{c}_2, \ \mathbf{c}_3\}$ be a basis for W. Let also

$$T : V \to W$$

be a linear transformation such that

$$T(\mathbf{b}_1) = 3\mathbf{c}_1 - 2\mathbf{c}_2 + 5\mathbf{c}_3, \qquad T(\mathbf{b}_2) = 4\mathbf{c}_1 + 7\mathbf{c}_2 - \mathbf{c}_3$$

Find the matrix M for the linear transformation T relative to the bases $\mathcal{B}$ and $\mathcal{C}$.

Solution: Let $\mathbf{x} = x_1\mathbf{b}_1 + x_2\mathbf{b}_2$, then

$$\begin{aligned} T(\mathbf{x}) &= T(x_1\mathbf{b}_1 + x_2\mathbf{b}_2) \\ &= T(x_1\mathbf{b}_1) + T(x_2\mathbf{b}_2) \\ &= x_1T(\mathbf{b}_1) + x_2T(\mathbf{b}_2) \\ &= x_1(3\mathbf{c}_1 - 2\mathbf{c}_2 + 5\mathbf{c}_3) + x_2(4\mathbf{c}_1 + 7\mathbf{c}_2 - \mathbf{c}_3) \\ &= (3x_1 + 4x_2)\mathbf{c}_1 + (-2x_1 + 7x_2)\mathbf{c}_2 + (5x_1 - x_2)\mathbf{c}_3 \end{aligned}$$

So,

$$T(\mathbf{x}) = (3x_1 + 4x_2)\mathbf{c}_1 + (-2x_1 + 7x_2)\mathbf{c}_2 + (5x_1 - x_2)\mathbf{c}_3$$

Hence

$$T(\mathbf{x}) = \begin{bmatrix} 3x_1 + 4x_2 \\ -2x_1 + 7x_2 \\ 5x_1 - x_2 \end{bmatrix} = \begin{bmatrix} 3 & 4 \\ -2 & 7 \\ 5 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Therefore $M = \begin{bmatrix} 3 & 4 \\ -2 & 7 \\ 5 & -1 \end{bmatrix}$.

Theorem 5.5 Let $\mathcal{V}$ and $\mathcal{W}$ be nontrivial vector spaces, with $\dim(\mathcal{V}) = n$ and $\dim(\mathcal{W}) = m$. Let $B = (\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n)$ and $C = (\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_m)$ be ordered bases for $\mathcal{V}$ and $\mathcal{W}$, respectively. Let $L: \mathcal{V} \to \mathcal{W}$ be a linear transformation. Then there is a unique $m \times n$ matrix $\mathbf{A}_{BC}$ such that $\mathbf{A}_{BC}[\mathbf{v}]_B = [L(\mathbf{v})]_C$, for all $\mathbf{v} \in \mathcal{V}$. (That is, $\mathbf{A}_{BC}$ times the coordinatization of $\mathbf{v}$ with respect to B gives the coordinatization of $L(\mathbf{v})$ with respect to C.)

Furthermore, for $1 \le i \le n$, the ith column of $\mathbf{A}_{BC} = [L(\mathbf{v}_i)]_C$.

Proof. Consider the $m \times n$ matrix $\mathbf{A}_{BC}$ whose ith column equals $[L(\mathbf{v}_i)]_C$, for $1 \le i \le n$. Let $\mathbf{v} \in \mathcal{V}$. We first prove that $\mathbf{A}_{BC}[\mathbf{v}]_B = [L(\mathbf{v})]_C$.

Suppose that $[\mathbf{v}]_B = [k_1, k_2, \ldots, k_n]$. Then $\mathbf{v} = k_1\mathbf{v}_1 + k_2\mathbf{v}_2 + \cdots + k_n\mathbf{v}_n$, and $L(\mathbf{v}) = k_1 L(\mathbf{v}_1) + k_2 L(\mathbf{v}_2) + \cdots + k_n L(\mathbf{v}_n)$, by Theorem 5.1. Hence,

$$\begin{aligned}
[L(\mathbf{v})]_C &= [k_1 L(\mathbf{v}_1) + k_2 L(\mathbf{v}_2) + \cdots + k_n L(\mathbf{v}_n)]_C \\
&= k_1 [L(\mathbf{v}_1)]_C + k_2 [L(\mathbf{v}_2)]_C + \cdots + k_n [L(\mathbf{v}_n)]_C \quad \text{by Theorem 4.19} \\
&= k_1(\text{1st column of } \mathbf{A}_{BC}) + k_2(\text{2nd column of } \mathbf{A}_{BC}) \\
&\quad + \cdots + k_n(n\text{th column of } \mathbf{A}_{BC}) \\
&= \mathbf{A}_{BC} \begin{bmatrix} k_1 \\ k_2 \\ \vdots \\ k_n \end{bmatrix} = \mathbf{A}_{BC}[\mathbf{v}]_B.
\end{aligned}$$

To complete the proof, we need to establish the uniqueness of $\mathbf{A}_{BC}$. Suppose that $\mathbf{H}$ is an $m \times n$ matrix such that $\mathbf{H}[\mathbf{v}]_B = [L(\mathbf{v})]_C$ for all $\mathbf{v} \in \mathcal{V}$. We will show that $\mathbf{H} = \mathbf{A}_{BC}$. It is enough to show that the ith column of $\mathbf{H}$ equals the ith column of $\mathbf{A}_{BC}$, for $1 \le i \le n$. Consider the ith vector, $\mathbf{v}_i$, of the ordered basis B for $\mathcal{V}$. Since $[\mathbf{v}_i]_B = \mathbf{e}_i$, we have ith column of $\mathbf{H} = \mathbf{H}\mathbf{e}_i = \mathbf{H}[\mathbf{v}_i]_B = [L(\mathbf{v}_i)]_C$, and this is the ith column of $\mathbf{A}_{BC}$. □

Example 3

We will find the matrix for the linear transformation $L: \mathcal{P}_3 \to \mathbb{R}^3$ given by

$$L(a_3x^3 + a_2x^2 + a_1x + a_0) = [a_0 + a_1, 2a_2, a_3 - a_0]$$

with respect to the standard ordered bases $B = (x^3, x^2, x, 1)$ for $\mathcal{P}_3$ and $C = (\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$ for $\mathbb{R}^3$. We first need to find $L(\mathbf{v})$, for each $\mathbf{v} \in B$. By definition of L, we have

$$L(x^3) = [0,0,1], \quad L(x^2) = [0,2,0], \quad L(x) = [1,0,0], \text{ and } \quad L(1) = [1,0,-1].$$

Since we are using the standard basis C for $\mathbb{R}^3$, each of these images in $\mathbb{R}^3$ is its own C-coordinatization. Then by Theorem 5.5, the matrix $\mathbf{A}_{BC}$ for L is the matrix whose columns are these images; that is,

$$\mathbf{A}_{BC} = \begin{array}{c} \begin{matrix} L(x^3) & L(x^2) & L(x) & L(1) \end{matrix} \\ \begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 2 & 0 & 0 \\ 1 & 0 & 0 & -1 \end{bmatrix} \end{array}.$$

We will compute $L(5x^3 - x^2 + 3x + 2)$ using this matrix. Now, $\left[5x^3 - x^2 + 3x + 2\right]_B = [5, -1, 3, 2]$. Hence, multiplication by $\mathbf{A}_{BC}$ gives

$$\left[L(5x^3 - x^2 + 3x + 2)\right]_C = \begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 2 & 0 & 0 \\ 1 & 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} 5 \\ -1 \\ 3 \\ 2 \end{bmatrix} = \begin{bmatrix} 5 \\ -2 \\ 3 \end{bmatrix}.$$

Since C is the standard basis for $\mathbb{R}^3$, we have $L(5x^3 - x^2 + 3x + 2) = [5, -2, 3]$, which can be quickly verified to be the correct answer. ■

Example 4

We will find the matrix for the same linear transformation $L: \mathcal{P}_3 \to \mathbb{R}^3$ of Example 3 with respect to the different ordered bases

$$D = (x^3 + x^2,\ x^2 + x,\ x + 1,\ 1)$$
$$\text{and} \quad E = ([-2,1,-3],[1,-3,0],[3,-6,2]).$$

You should verify that D and E are bases for $\mathcal{P}_3$ and $\mathbb{R}^3$, respectively.

We first need to find $L(\mathbf{v})$, for each $\mathbf{v} \in D$. By definition of L, we have $L(x^3 + x^2) = [0,2,1]$, $L(x^2 + x) = [1,2,0]$, $L(x + 1) = [2,0,-1]$, and $L(1) = [1,0,-1]$. Now we must find the coordinatization of each of these images in terms of the basis E for $\mathbb{R}^3$. Since we must solve for the coordinates of many vectors, it is quicker to use the transition matrix $\mathbf{Q}$ from the standard basis C for $\mathbb{R}^3$ to the basis E. From Theorem 4.22, $\mathbf{Q}$ is the inverse of the matrix whose columns are the vectors in E; that is,

$$\mathbf{Q} = \begin{bmatrix} -2 & 1 & 3 \\ 1 & -3 & -6 \\ -3 & 0 & 2 \end{bmatrix}^{-1} = \begin{bmatrix} -6 & -2 & 3 \\ 16 & 5 & -9 \\ -9 & -3 & 5 \end{bmatrix}.$$

Now, multiplying $\mathbf{Q}$ by each of the images, we get

$$\left[L(x^3 + x^2)\right]_E = \mathbf{Q}\begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \\ -1 \end{bmatrix}, \qquad \left[L(x^2 + x)\right]_E = \mathbf{Q}\begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} = \begin{bmatrix} -10 \\ 26 \\ -15 \end{bmatrix},$$

$$[L(x+1)]_E = \mathbf{Q}\begin{bmatrix} 2 \\ 0 \\ -1 \end{bmatrix} = \begin{bmatrix} -15 \\ 41 \\ -23 \end{bmatrix}, \quad \text{and} \quad [L(1)]_E = \mathbf{Q}\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} = \begin{bmatrix} -9 \\ 25 \\ -14 \end{bmatrix}.$$

By Theorem 5.5, the matrix $\mathbf{A}_{DE}$ for L is the matrix whose columns are these products.

$$\mathbf{A}_{DE} = \begin{bmatrix} -1 & -10 & -15 & -9 \\ 1 & 26 & 41 & 25 \\ -1 & -15 & -23 & -14 \end{bmatrix}$$

We will compute $L(5x^3 - x^2 + 3x + 2)$ using this matrix. We must first find the representation for $5x^3 - x^2 + 3x + 2$ in terms of the basis D. Solving $5x^3 - x^2 + 3x + 2 = a(x^3 + x^2) + b(x^2 + x) + c(x + 1) + d(1)$ for $a, b, c,$ and d, we get the unique solution $a = 5$, $b = -6$, $c = 9$, and $d = -7$ (verify!). Hence, $\left[5x^3 - x^2 + 3x + 2\right]_D = [5,-6,9,-7]$. Then

$$\left[L(5x^3 - x^2 + 3x + 2)\right]_E = \begin{bmatrix} -1 & -10 & -15 & -9 \\ 1 & 26 & 41 & 25 \\ -1 & -15 & -23 & -14 \end{bmatrix}\begin{bmatrix} 5 \\ -6 \\ 9 \\ -7 \end{bmatrix} = \begin{bmatrix} -17 \\ 43 \\ -24 \end{bmatrix}.$$

This answer represents a coordinate vector in terms of the basis E, and so

$$L(5x^3 - x^2 + 3x + 2) = -17\begin{bmatrix} -2 \\ 1 \\ -3 \end{bmatrix} + 43\begin{bmatrix} 1 \\ -3 \\ 0 \end{bmatrix} - 24\begin{bmatrix} 3 \\ -6 \\ 2 \end{bmatrix} = \begin{bmatrix} 5 \\ -2 \\ 3 \end{bmatrix},$$

which agrees with the answer in Example 3. ■