Coordinatization

DEFINITION: Suppose $B = \{\mathbf{b}_1, \ldots, \mathbf{b}_n\}$ is a basis for a vector space V and $\mathbf{x}$ is in V. The **coordinates** of $\mathbf{x}$ relative to the basis B are the weights $c_1, \ldots, c_n$ such that

$$\mathbf{x} = c_1\mathbf{b}_1 + \ldots + c_n\mathbf{b}_n$$

NOTATION:

$$[\mathbf{x}]_B = \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}$$

THEOREM: Let $B = \{\mathbf{b}_1, \ldots, \mathbf{b}_n\}$ be a basis for a vector space V. Then for each $\mathbf{x}$ in V, there exists a unique set of scalars $c_1, \ldots, c_n$ such that

$$\mathbf{x} = c_1\mathbf{b}_1 + \ldots + c_n\mathbf{b}_n$$

EXAMPLE: Let

$$\mathbf{b}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad \mathbf{b}_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \quad \mathbf{x} = \begin{bmatrix} 1 \\ 6 \end{bmatrix}$$

Find coordinates of $\mathbf{x}$ in $\{\mathbf{b}_1, \ \mathbf{b}_2\}$.

Solution: We have

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 6 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 3 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 3 \end{bmatrix}$$

therefore

$$c_1 = -2 \quad \text{and} \quad c_2 = 3$$

so

$$[\mathbf{x}]_B = \begin{bmatrix} -2 \\ 3 \end{bmatrix}$$

EXAMPLE: Let $B = \{1, \ t, \ t^2\}$ be the standard basis for $\mathbb{P}_2$. Find coordinates of the vector

$$\mathbf{p}(t) = -4 + 3t - 5t^2$$

relative to B.

Solution: By the definition above we have

$$[\mathbf{p}]_B = \begin{bmatrix} -4 \\ 3 \\ -5 \end{bmatrix}$$

EXAMPLE: Determine whether the polynomials

$$1 + t, \quad 1 + t^2, \quad t + t^2$$

form a basis for $\mathbb{P}_2$. If "Yes", find coordinates of the vector

$$\mathbf{p}(t) = -4 + 3t - 5t^2$$

relative to this basis.

Solution: Let $B = \{1,\ t,\ t^2\}$ be the standard basis of $\mathbb{P}_2$. Then the polynomials

$$1+t, \quad 1+t^2, \quad t+t^2$$

produce coordinate vectors

$$\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \quad \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

relative to B. We have:

$$\begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & 2 \end{bmatrix}$$

Since there are 3 pivots and 3 columns, the polynomials above form a basis for $\mathbb{P}_2$.

Let

$$B = \{1+t,\ 1+t^2,\ t+t^2\}$$

To find coordinates of the vector

$$\mathbf{p}(t) = -4 + 3t - 5t^2$$

relative to B, we consider the augmented matrix

$$\begin{bmatrix} 1 & 1 & 0 & -4 \\ 1 & 0 & 1 & 3 \\ 0 & 1 & 1 & -5 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & -6 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

therefore

$$[\mathbf{p}]_B = \begin{bmatrix} 2 \\ -6 \\ 1 \end{bmatrix}$$

EXAMPLE: Consider the following 2 bases of the vector space $\mathbb{R}^3$

$$\mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{e}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad \mathbf{e}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

and

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

Find coordinates of the vector

$$\mathbf{x} = \langle -4, 3, -5 \rangle$$

in $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ and in $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$.

Solution:

(a) Let

$$S = \{\mathbf{e}_1,\ \mathbf{e}_2,\ \mathbf{e}_3\}$$

To find coordinates of the vector

$$\mathbf{x} = \langle -4, 3, -5 \rangle$$

relative to S, we consider the augmented matrix

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & -4 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & -5 \end{array}\right]$$

therefore

$$[\mathbf{x}]_S = \begin{bmatrix} -4 \\ 3 \\ -5 \end{bmatrix}$$

(b) Let

$$B = \{\mathbf{v}_1,\ \mathbf{v}_2,\ \mathbf{v}_3\}$$

To find coordinates of the vector

$$\mathbf{x} = \langle -4, 3, -5 \rangle$$

relative to B, we consider the augmented matrix

$$\left[\begin{array}{ccc|c} 1 & 1 & 0 & -4 \\ 1 & 0 & 1 & 3 \\ 0 & 1 & 1 & -5 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & -6 \\ 0 & 0 & 1 & 1 \end{array}\right]$$

therefore

$$[\mathbf{x}]_B = \begin{bmatrix} 2 \\ -6 \\ 1 \end{bmatrix}$$

<u>Conclusion</u>: The vector

$$\mathbf{x} = \langle -4, 3, -5 \rangle$$

has two different coordinates in two different bases:

$$[\mathbf{x}]_S = \begin{bmatrix} -4 \\ 3 \\ -5 \end{bmatrix} \quad \text{and} \quad [\mathbf{x}]_B = \begin{bmatrix} 2 \\ -6 \\ 1 \end{bmatrix}$$

where

$$S = \{\mathbf{e}_1,\ \mathbf{e}_2,\ \mathbf{e}_3\}$$

and

$$B = \{\mathbf{v}_1,\ \mathbf{v}_2,\ \mathbf{v}_3\}$$

THEOREM 4.20: Let $B = \{\mathbf{b}_1, \ldots, \mathbf{b}_n\}$ and $C = \{\mathbf{c}_1, \ldots, \mathbf{c}_n\}$ be ordered bases of a vector space V. Then there is a unique matrix P (called a **transition matrix**) such that

$$[\mathbf{x}]_C = P[\mathbf{x}]_B$$

where

$$P = [[\mathbf{b}_1]_C \ \ [\mathbf{b}_2]_C \ \ \ldots \ \ [\mathbf{b}_n]_C]$$

EXAMPLE: Let $\mathbf{x} = \langle -4, 3, -5 \rangle$, $S = \{\mathbf{e}_1,\ \mathbf{e}_2,\ \mathbf{e}_3\}$ and $B = \{\mathbf{v}_1,\ \mathbf{v}_2,\ \mathbf{v}_3\}$, then

$$[\mathbf{x}]_S = \begin{bmatrix} -4 \\ 3 \\ -5 \end{bmatrix}, \quad [\mathbf{x}]_B = \begin{bmatrix} 2 \\ -6 \\ 1 \end{bmatrix}$$

$$[\mathbf{v}_1]_S = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \quad [\mathbf{v}_2]_S = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \quad [\mathbf{v}_3]_S = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

therefore

$$P = [[\mathbf{v}_1]_S \ \ [\mathbf{v}_2]_S \ \ \dots \ \ [\mathbf{v}_n]_S] = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

and in fact

$$\begin{bmatrix} -4 \\ 3 \\ -5 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ -6 \\ 1 \end{bmatrix}$$

THEOREM 4.21: Suppose that B, C, and D are ordered bases for a nontrivial finite dimensional vector space V. Let P be the transition matrix from B to C, and let Q be the transition matrix from C to D. Then QP is the transition matrix from B to D.

THEOREM 4.22: Let B and C be ordered bases for a nontrivial finite dimensional vector space V, and let P be the transition matrix from B to C. Then P is nonsingular, and P^{-1} is the transition matrix from C to B.

EXAMPLE: We have $P = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$ and

$$[\mathbf{x}]_S = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} [\mathbf{x}]_B$$

therefore

$$[\mathbf{x}]_B = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}^{-1} [\mathbf{x}]_S$$

We have

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & -1 & 1 & -1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & -1 & 1 & -1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 1 & -1 & 0 \\ 0 & 0 & 2 & -1 & 1 & 1 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1/2 & 1/2 & 1/2 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1/2 & 1/2 & -1/2 \\ 0 & 1 & 0 & 1/2 & -1/2 & 1/2 \\ 0 & 0 & 1 & -1/2 & 1/2 & 1/2 \end{array}\right]$$

So

$$\begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1/2 & 1/2 & -1/2 \\ 1/2 & -1/2 & 1/2 \\ -1/2 & 1/2 & 1/2 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 1 & 1 & -1 \\ 1 & -1 & 1 \\ -1 & 1 & 1 \end{bmatrix}$$

therefore

$$[\mathbf{x}]_B = \frac{1}{2}\begin{bmatrix} 1 & 1 & -1 \\ 1 & -1 & 1 \\ -1 & 1 & 1 \end{bmatrix} [\mathbf{x}]_S$$

EXAMPLE: Let $B = \{\mathbf{b}_1, \mathbf{b}_2\}$ and $C = \{\mathbf{c}_1, \mathbf{c}_2\}$ be bases for a vector space V, such that

$$\mathbf{b}_1 = 4\mathbf{c}_1 + \mathbf{c}_2 \qquad \text{and} \qquad \mathbf{b}_2 = -6\mathbf{c}_1 + \mathbf{c}_2$$

Suppose

$$\mathbf{x} = 3\mathbf{b}_1 + \mathbf{b}_2$$

Find $[\mathbf{x}]_C$.

Solution 1: We have

$$\begin{aligned} \mathbf{x} = 3\mathbf{b}_1 + \mathbf{b}_2 &= 3(4\mathbf{c}_1 + \mathbf{c}_2) + (-6\mathbf{c}_1 + \mathbf{c}_2) \\ &= 12\mathbf{c}_1 + 3\mathbf{c}_2 - 6\mathbf{c}_1 + \mathbf{c}_2 \\ &= 6\mathbf{c}_1 + 4\mathbf{c}_2 \end{aligned}$$

therefore $[\mathbf{x}]_C = \langle 6, 4 \rangle$.

Solution 2: We have $[\mathbf{x}]_B = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$ and

$$[\mathbf{b}_1]_C = \begin{bmatrix} 4 \\ 1 \end{bmatrix}, \quad [\mathbf{b}_2]_C = \begin{bmatrix} -6 \\ 1 \end{bmatrix}$$

therefore

$$P = \begin{bmatrix} 4 & -6 \\ 1 & 1 \end{bmatrix}$$

hence

$$[\mathbf{x}]_C = \begin{bmatrix} 4 & -6 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 6 \\ 4 \end{bmatrix}$$

EXAMPLE: Let

$$\mathbf{b}_1 = \begin{bmatrix} 1 \\ -3 \end{bmatrix}, \quad \mathbf{b}_2 = \begin{bmatrix} -2 \\ 4 \end{bmatrix}, \quad \mathbf{c}_1 = \begin{bmatrix} -7 \\ 9 \end{bmatrix}, \quad \mathbf{c}_2 = \begin{bmatrix} -5 \\ 7 \end{bmatrix}$$

and consider the bases for $\mathbb{R}^2$ given by $B = \{\mathbf{b}_1, \mathbf{b}_2\}$ and $C = \{\mathbf{c}_1, \mathbf{c}_2\}$.

(a) Find the transition matrix from B to C.

(b) Find the transition matrix from C to B.

Solution: The transition matrix from B to C involves the C-coordinate vectors of $\mathbf{b}_1$ and $\mathbf{b}_2$:

$$[\mathbf{x}]_C = [[\mathbf{b}_1]_C \;\; [\mathbf{b}_2]_C]\, [\mathbf{x}]_B$$

Let

$$[\mathbf{b}_1]_C = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \quad \text{and} \quad [\mathbf{b}_2]_C = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

Then, by definition,

$$x_1\mathbf{c}_1 + x_2\mathbf{c}_2 = \mathbf{b}_1 \quad \text{and} \quad y_1\mathbf{c}_1 + y_2\mathbf{c}_2 = \mathbf{b}_2$$

hence

$$[\mathbf{c}_1 \ \ \mathbf{c}_2]\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \mathbf{b}_1 \quad \text{and} \quad [\mathbf{c}_1 \ \ \mathbf{c}_2]\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \mathbf{b}_2$$

that is,

$$\begin{bmatrix} -7 & -5 \\ 9 & 7 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ -3 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} -7 & -5 \\ 9 & 7 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} -2 \\ 4 \end{bmatrix}$$

To solve each system of equations we write the corresponding augmented matrices and come up with the corresponding reduced echelon forms:

$$\left[\begin{array}{cc|c} -7 & -5 & 1 \\ 9 & 7 & -3 \end{array}\right] \sim \left[\begin{array}{cc|c} -7 & -5 & 1 \\ 2 & 2 & -2 \end{array}\right] \sim \left[\begin{array}{cc|c} -7 & -5 & 1 \\ 1 & 1 & -1 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 1 & -1 \\ -7 & -5 & 1 \end{array}\right]$$

$$\sim \left[\begin{array}{cc|c} 1 & 1 & -1 \\ 0 & 2 & -6 \end{array}\right]$$

$$\sim \left[\begin{array}{cc|c} 1 & 1 & -1 \\ 0 & 1 & -3 \end{array}\right]$$

$$\sim \left[\begin{array}{cc|c} 1 & 0 & 2 \\ 0 & 1 & -3 \end{array}\right]$$

and

$$\left[\begin{array}{cc|c} -7 & -5 & -2 \\ 9 & 7 & 4 \end{array}\right] \sim \left[\begin{array}{cc|c} -7 & -5 & -2 \\ 2 & 2 & 2 \end{array}\right] \sim \left[\begin{array}{cc|c} -7 & -5 & -2 \\ 1 & 1 & 1 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 1 & 1 \\ -7 & -5 & -2 \end{array}\right]$$

$$\sim \left[\begin{array}{cc|c} 1 & 1 & 1 \\ 0 & 2 & 5 \end{array}\right]$$

$$\sim \left[\begin{array}{cc|c} 1 & 1 & 1 \\ 0 & 1 & 5/2 \end{array}\right]$$

$$\sim \left[\begin{array}{cc|c} 1 & 0 & -3/2 \\ 0 & 1 & 5/2 \end{array}\right]$$

Note that we can solve both systems simultaneously augmenting the coefficient matrix with $\mathbf{b}_1$ and $\mathbf{b}_2$:

$$\left[\begin{array}{cc|cc} -7 & -5 & 1 & -2 \\ 9 & 7 & -3 & 4 \end{array}\right] \sim \left[\begin{array}{cc|cc} -7 & -5 & 1 & -2 \\ 2 & 2 & -2 & 2 \end{array}\right] \sim \left[\begin{array}{cc|cc} -7 & -5 & 1 & -2 \\ 1 & 1 & -1 & 1 \end{array}\right] \sim \left[\begin{array}{cc|cc} 1 & 1 & -1 & 1 \\ -7 & -5 & 1 & -2 \end{array}\right]$$

$$\sim \left[\begin{array}{cc|cc} 1 & 1 & -1 & 1 \\ 0 & 2 & -6 & 5 \end{array}\right]$$

$$\sim \left[\begin{array}{cc|cc} 1 & 1 & -1 & 1 \\ 0 & 1 & -3 & 5/2 \end{array}\right]$$

$$\sim \left[\begin{array}{cc|cc} 1 & 0 & 2 & -3/2 \\ 0 & 1 & -3 & 5/2 \end{array}\right]$$

So, the transition matrix from B to C is

$$\begin{bmatrix} 2 & -3/2 \\ -3 & 5/2 \end{bmatrix}$$

<u>Conclusion</u>: To find the transition matrix from B to C we start with the matrix

$$[\mathbf{c}_1\ \mathbf{c}_2\ \mathbf{b}_1\ \mathbf{b}_2]$$

and use elementary row operations to come up with

$$[I\ P_1]$$

where I is an identity matrix and P_1 is the transition matrix from B to C.

Similarly, to find the transition matrix from C to B we start with

$$[\mathbf{b}_1\ \mathbf{b}_2\ \mathbf{c}_1\ \mathbf{c}_2]$$

and use elementary row operations to come up with

$$[I\ P_2]$$

where I is an identity matrix and P_2 is the transition matrix from C to B ($P_2 = P_1^{-1}$ by Theorem 4.22 above). For example, since

$$\left[\begin{array}{cc|cc} 1 & -2 & -7 & -5 \\ -3 & 4 & 9 & 7 \end{array}\right] \sim \left[\begin{array}{cc|cc} 1 & -2 & -7 & -5 \\ 0 & -2 & -12 & -8 \end{array}\right] \sim \left[\begin{array}{cc|cc} 1 & -2 & -7 & -5 \\ 0 & 1 & 6 & 4 \end{array}\right] \sim \left[\begin{array}{cc|cc} 1 & 0 & 5 & 3 \\ 0 & 1 & 6 & 4 \end{array}\right]$$

it follows that the transition matrix from C to B is

$$\begin{bmatrix} 5 & 3 \\ 6 & 4 \end{bmatrix}$$

<u>Diagonalization and the Transition Matrix</u>: When the Diagonalization Method of Section 3.4 is successfully performed on a matrix A, the matrix P obtained is the transition matrix from B-coordinates to standard coordinates, where B is an ordered basis for $\mathbb{R}^n$ consisting of eigenvectors for A.

EXAMPLE: Let

$$A = \begin{bmatrix} 1 & 3 & 3 \\ -3 & -5 & -3 \\ 3 & 3 & 1 \end{bmatrix}$$

(a) Find the eigenvalues.

Solution: We first solve the following equation

$$\det(A - \lambda I) = \begin{vmatrix} 1-\lambda & 3 & 3 \\ -3 & -5-\lambda & -3 \\ 3 & 3 & 1-\lambda \end{vmatrix} = 0$$

Expanding this determinant, we obtain

$$-\lambda^3 - 3\lambda^2 + 4 = (1-\lambda)(\lambda+2)^2 = 0$$

hence

$$\lambda_1 = 1, \quad \lambda_2 = -2$$

are the eigenvalues of A

(b) Find the corresponding eigenvectors.

Solution:
Let $\lambda = 1$. We have

$$\begin{bmatrix} 1-\lambda & 3 & 3 & 0 \\ -3 & -5-\lambda & -3 & 0 \\ 3 & 3 & 1-\lambda & 0 \end{bmatrix} = \begin{bmatrix} 0 & 3 & 3 & 0 \\ -3 & -6 & -3 & 0 \\ 3 & 3 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 2 & 1 & 0 \\ 1 & 1 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 2 & 1 & 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

hence

$$\begin{cases} x_1 - x_3 = 0 \\ x_2 + x_3 = 0 \end{cases} \implies \begin{cases} x_1 = x_3 \\ x_2 = -x_3 \end{cases}$$

therefore

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_3 \\ -x_3 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

is the eigenvector of A, where x_3 is any nonzero scalar. In particular, $\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$ is an eigenvector of A.

Let $\lambda = -2$. We have

$$\begin{bmatrix} 1-\lambda & 3 & 3 & 0 \\ -3 & -5-\lambda & -3 & 0 \\ 3 & 3 & 1-\lambda & 0 \end{bmatrix} = \begin{bmatrix} 3 & 3 & 3 & 0 \\ -3 & -3 & -3 & 0 \\ 3 & 3 & 3 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

hence

$$x_1 + x_2 + x_3 = 0 \quad \implies \quad x_1 = -x_2 - x_3$$

therefore

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -x_2 - x_3 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} (-1) \cdot x_2 & + & (-1) \cdot x_3 \\ 1 \cdot x_2 & + & 0 \cdot x_3 \\ 0 \cdot x_2 & + & 1 \cdot x_3 \end{bmatrix} = x_2 \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

is the eigenvector of A, where x_2, x_3 are any scalars (not both zero). In particular, $\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$, $\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$ are eigenvectors of A.

(c) Diagonalize A.

Solution: By (b),

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

are eigenvectors of A. Moreover,

$$\begin{vmatrix} 1 & -1 & -1 \\ -1 & 1 & 0 \\ 1 & 0 & 1 \end{vmatrix} \neq 0$$

Therefore

$$P = \begin{bmatrix} 1 & -1 & -1 \\ -1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}, \quad D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{bmatrix}$$

and

$$\begin{bmatrix} 1 & 3 & 3 \\ -3 & -5 & -3 \\ 3 & 3 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -1 & -1 \\ -1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{bmatrix} \begin{bmatrix} 1 & -1 & -1 \\ -1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}^{-1}$$

Let

$$B = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$$

Note that B is a basis for $\mathbb{R}^3$, because $\det([\mathbf{v}_1 \ \mathbf{v}_2 \ \mathbf{v}_3]) \neq 0$. Since

$$P = [[\mathbf{v}_1]_S \quad [\mathbf{v}_2]_S \quad [\mathbf{v}_3]_S]$$

where S is the standard basis of $\mathbb{R}^3$, it follows that P is the transition matrix from B-coordinates to standard coordinates. Moreover, by Theorem 4.22, P^{-1} is the transition matrix from standard coordinates to B-coordinates. Furthermore, since $A = PDP^{-1}$, it follows that $P^{-1}AP = D$. We can understand the relationship between A and D more fully from a "change of coordinates" perspective. In fact, if $\mathbf{v}$ is any vector in $\mathbb{R}^3$ expressed in standard coordinates, we claim that

$$D[\mathbf{v}]_B = [A\mathbf{v}]_B$$

That is, multiplication by D when working in B-coordinates corresponds to first multiplying by A in standard coordinates, and then converting the result to B-coordinates. Here is why does this relationship holds:

$$D[\mathbf{v}]_B = (P^{-1}AP)[\mathbf{v}]_B = (P^{-1}A)P[\mathbf{v}]_B = P^{-1}A[\mathbf{v}]_S = P^{-1}(A\mathbf{v}) = [A\mathbf{v}]_B$$

because multiplication by P and P^{-1} performs the appropriate transitions between B- and S-coordinates. Thus, we can think of D as being the "B-coordinates version" of A. By using a basis of eigenvectors we have converted to a new coordinate system in which multiplication by A has been replaced with multiplication by a diagonal matrix, which is much easier to work with because of its simpler form.