Constructing Special Bases

THEOREM: The set of vectors $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ is a basis of $\mathbb{R}^n$ if and only if $n = p$ and the matrix $A = [\mathbf{v}_1 \ldots \mathbf{v}_p]$ has exactly n pivot positions.

EXAMPLE: Let

$$\mathbf{v}_1 = \begin{bmatrix} 3 \\ 0 \\ -6 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} -4 \\ 1 \\ 7 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} -2 \\ 1 \\ 5 \end{bmatrix}$$

Determine if $\{\mathbf{v}_1,\ \mathbf{v}_2,\ \mathbf{v}_3\}$ is a basis for $\mathbb{R}^3$.

Solution: We have

$$\begin{bmatrix} 3 & -4 & -2 \\ 0 & 1 & 1 \\ -6 & 7 & 5 \end{bmatrix} \sim \begin{bmatrix} 3 & -4 & -2 \\ 0 & 1 & 1 \\ 0 & -1 & 1 \end{bmatrix} \sim \begin{bmatrix} 3 & -4 & -2 \\ 0 & 1 & 1 \\ 0 & 0 & 2 \end{bmatrix}$$

Since we have 3 vectors and 3 pivots, $\{\mathbf{v}_1,\ \mathbf{v}_2,\ \mathbf{v}_3\}$ is a basis for $\mathbb{R}^3$.

THEOREM: Let $\mathbf{v}_1, \ldots, \mathbf{v}_m$ be vectors from $\mathbb{R}^n$. The pivot columns of the matrix $A = [\mathbf{v}_1 \ \ldots \ \mathbf{v}_m]$ form a basis for $\text{span}(\{\mathbf{v}_1, \ldots, \mathbf{v}_m\})$.

EXAMPLE: It can be shown that the matrix

$$A = [\mathbf{v}_1\ \mathbf{v}_2\ \mathbf{v}_3\ \mathbf{v}_4\ \mathbf{v}_5] = \begin{bmatrix} 1 & 4 & 0 & 2 & -1 \\ 3 & 12 & 1 & 5 & 5 \\ 2 & 8 & 1 & 3 & 2 \\ 5 & 20 & 2 & 8 & 8 \end{bmatrix}$$

is row equivalent to the matrix

$$\begin{bmatrix} 1 & 4 & 0 & 2 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Find the bases for $\text{span}(\{\mathbf{v}_1, \ldots, \mathbf{v}_5\})$.

Solution: By the Theorem above, $\{\mathbf{v}_1,\ \mathbf{v}_3,\ \mathbf{v}_5\}$ is a basis for $\text{span}(\{\mathbf{v}_1, \ldots, \mathbf{v}_5\})$.

EXAMPLE: Let

$$\mathbf{v}_1 = \begin{bmatrix} 3 \\ 0 \\ -6 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} -4 \\ 1 \\ 7 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} -2 \\ 1 \\ 3 \end{bmatrix}$$

Find a basis for $\text{span}(\{\mathbf{v}_1,\ \mathbf{v}_2,\ \mathbf{v}_3\})$.

Solution 1: We have

$$\begin{bmatrix} 3 & -4 & -2 \\ 0 & 1 & 1 \\ -6 & 7 & 3 \end{bmatrix} \sim \begin{bmatrix} 3 & -4 & -2 \\ 0 & 1 & 1 \\ 0 & -1 & -1 \end{bmatrix} \sim \begin{bmatrix} 3 & -4 & -2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

Since the first and the second columns are pivot columns, $\{\mathbf{v}_1,\ \mathbf{v}_2\}$ is a basis for $\text{span}(\{\mathbf{v}_1,\ \mathbf{v}_2,\ \mathbf{v}_3\})$.

Solution 2: Now let use the Simplified Span Method. We have

$$\begin{bmatrix} 3 & 0 & -6 \\ -4 & 1 & 7 \\ -2 & 1 & 3 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -2 \\ -4 & 1 & 7 \\ -2 & 1 & 3 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & -1 \\ 0 & 1 & -1 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

It follows that

$$\begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix}, \quad \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$$

is a basis for $\text{span}(\{\mathbf{v}_1, \ \mathbf{v}_2, \ \mathbf{v}_3\})$.

EXAMPLE: Let

$$Y = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} \right\}$$

Find a basis for $\text{span}(Y)$.

Solution: We express the matrices in Y as corresponding vectors in $\mathbb{R}^4$ and create the matrix with these vectors as columns, as follows:

$$A = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 \end{bmatrix}, \quad \text{which reduces to} \quad A = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Then, the desired basis is

$$\left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \right\}$$

THEOREM 4.17: If S is a spanning set for a finite dimensional vector space V, then there is a set $B \subseteq S$ that is a basis for V.