Basis and Dimension

DEFINITION: Let V be a vector space, and let B be a subset of V. Then B is a **basis** for V if and only if both of the following are true:

(1) B spans V.

(2) B is linearly independent.

EXAMPLE: Show that the vectors

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} -2 \\ -1 \\ -4 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} 3 \\ 5 \\ 7 \end{bmatrix}$$

form a basis for $\mathbb{R}^3$.

Solution: We have

$$[\mathbf{v}_1\ \mathbf{v}_2\ \mathbf{v}_3] = \begin{bmatrix} 1 & -2 & 3 \\ 1 & -1 & 5 \\ 2 & -4 & 7 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$$

Since the number of pivots is equal to the number of columns, the vectors are linearly independent. Since the number of pivots is equal to the number of rows, the vectors span $\mathbb{R}^3$. Therefore $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ form a basis for $\mathbb{R}^3$.

EXAMPLE: Show that the vectors (polynomials)

$$\mathbf{p}_1 = 1 + t, \quad \mathbf{p}_2 = 1 + t^2, \quad \mathbf{p}_3 = t + t^2$$

form a basis for $\mathbb{P}_2$.

Solution: We have

$$\begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & 2 \end{bmatrix}$$

Since the number of pivots is equal to the number of columns, the vectors are linearly independent. Since the number of pivots is equal to the number of rows, the vectors span $\mathbb{P}_2$. Therefore $\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3$ form a basis for $\mathbb{P}_2$.

STANDARD BASIS FOR $\mathbb{R}^n$:

$$\mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \quad \mathbf{e}_2 = \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix}, \quad \ldots, \quad \mathbf{e}_n = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}$$

STANDARD BASIS FOR $\mathbb{P}_n$:

$$\mathbf{p}_1 = 1, \quad \mathbf{p}_2 = t, \quad \mathbf{p}_3 = t^2, \quad \ldots, \quad \mathbf{p}_{n+1} = t^n$$

form the so-called standard basis for the vector space $\mathbb{P}_n$.

LEMMA 4.11: Let S and T be subsets of a vector space V such that S spans V, S is finite, and T is linearly independent. Then T is finite and $|T| \leq |S|$.

Dimension

THEOREM 4.12: Let V be a vector space, and let B_1 and B_2 be bases for V such that B_1 has finitely many elements. Then B_2 also has finitely many elements, and $|B_1| = |B_2|$.

DEFINITION: Let V be a vector space. If V has a basis B containing a finite number of elements, then V is said to be **finite dimensional**. In this case, the **dimension** of V, $\dim(V)$, is the number of elements in any basis for V. In particular, $\dim(V) = |B|$. If V has no finite basis, then V is **infinite dimensional**.

EXAMPLE: The dimension of $\mathbb{R}^n$ is n, the dimension of $\mathbb{P}_n$ is $n+1$, the dimension of M_{mn} is mn.

EXAMPLE: Let $V = \{\mathbf{0}\}$ be the trivial vector space. Then $\dim(V) = 0$ because the set $\{\mathbf{0}\}$ is linearly dependent, since the equation $c\mathbf{0} = \mathbf{0}$ has a nontrivial solution.

Sizes of Spanning Sets and Linearly Independent Sets

THEOREM 4.13: Let V be a finite dimensional vector space.

(1) Suppose S is a finite subset of V that spans V. Then $\dim(V) \leq |S|$. Moreover, $|S| = \dim(V)$ if and only if S is a basis for V.

(2) Suppose T is a linearly independent subset of V. Then T is finite and $|T| \leq \dim(V)$. Moreover, $|T| = \dim(V)$ if and only if T is a basis for V.

THEOREM 4.14: Let V be a vector space with spanning set S (so, $\text{span}(S) = V$), and let B be a maximal linearly independent subset of S. Then B is a basis for V.

REMARK: The phrase "B is a maximal linearly independent subset of S" means that both of the following are true:

- B is a linearly independent subset of S.
- If $B \subset C \subseteq S$ and $B \neq C$, then C is linearly dependent.

THEOREM 4.15: Let V be a vector space, and let B be a minimal spanning set for V. Then B is a basis for V.

REMARK: The phrase "B is a minimal spanning set for V" means that both of the following are true:

- B is a subset of V that spans V.
- If $C \subset B$ and $C \neq B$, then C does not span V.

Dimension of a Subspace

THEOREM 4.16: Let V be a finite dimensional vector space, and let W be a subspace of V. Then W is also finite dimensional with $\dim(W) \leq \dim(V)$. Moreover, $\dim(W) = \dim(V)$ if and only if $W = V$.