Linear Independence

DEFINITION: Vectors $\mathbf{v}_1, \ldots, \mathbf{v}_p$ are said to be linearly **dependent** if there exist scalars $c_1, \ldots, c_p$, not all zero, such that

$$c_1\mathbf{v}_1 + \ldots + c_p\mathbf{v}_p = \mathbf{0}$$

Vectors $\mathbf{v}_1, \ldots, \mathbf{v}_p$ are said to be linearly **independent** if the vector equation

$$c_1\mathbf{v}_1 + \ldots + c_p\mathbf{v}_p = \mathbf{0}$$

has only the trivial solution.

EXAMPLE: Show that the vectors

$$\mathbf{u}_1 = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}, \quad \mathbf{u}_2 = \begin{bmatrix} -2 \\ -1 \\ -4 \end{bmatrix}, \quad \mathbf{u}_3 = \begin{bmatrix} 3 \\ 5 \\ 6 \end{bmatrix}$$

are linearly dependent. Then show that the vectors

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} -2 \\ -1 \\ -4 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} 3 \\ 5 \\ 7 \end{bmatrix}$$

are linearly independent.

Solution: To show that the vectors $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$ are linearly dependent, we find c_1, c_2, c_3, not all zero, such that

$$c_1\mathbf{u}_1 + c_2\mathbf{u}_2 + c_3\mathbf{u}_3 = \mathbf{0}$$

that is,

$$c_1 \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} + c_2 \begin{bmatrix} -2 \\ -1 \\ -4 \end{bmatrix} + c_3 \begin{bmatrix} 3 \\ 5 \\ 6 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad \Longleftrightarrow \quad \begin{bmatrix} c_1 - 2c_2 + 3c_3 \\ c_1 - c_2 + 5c_3 \\ 2c_1 - 4c_2 + 6c_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

This can be rewritten as the following system of equations:

$$\begin{cases} c_1 - 2c_2 + 3c_3 = 0 \\ c_1 - c_2 + 5c_3 = 0 \\ 2c_1 - 4c_2 + 6c_3 = 0 \end{cases} \Longrightarrow \begin{cases} c_1 - 2c_2 + 3c_3 = 0 \\ c_2 + 2c_3 = 0 \end{cases} \Longrightarrow \begin{cases} c_1 + 7c_3 = 0 \\ c_2 + 2c_3 = 0 \end{cases} \Longrightarrow \begin{cases} c_1 = -7c_3 \\ c_2 = -2c_3 \\ c_3 \text{ is free} \end{cases}$$

For example, if $c_3 = -1$, then $c_1 = 7$ and $c_2 = 2$, that is,

$$7\mathbf{u}_1 + 2\mathbf{u}_2 - \mathbf{u}_3 = \mathbf{0}$$

We now show that

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3 = \mathbf{0}$$

has only the trivial solution (which means that the vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ are linearly independent). We have

$$\begin{cases} c_1 - 2c_2 + 3c_3 = 0 \\ c_1 - c_2 + 5c_3 = 0 \\ 2c_1 - 4c_2 + 6c_3 = 0 \end{cases} \Longrightarrow \begin{cases} c_1 - 2c_2 + 3c_3 = 0 \\ c_2 + 2c_3 = 0 \\ c_3 = 0 \end{cases} \Longrightarrow \begin{cases} c_1 = 0 \\ c_2 = 0 \\ c_3 = 0 \end{cases}$$

THEOREM: Let A be an $m \times n$ matrix. Then the following statements are logically equivalent:

1. A homogeneous system $A\mathbf{x} = \mathbf{0}$ has only the trivial solution.
2. There are no free variables.
3. Number of columns of $A =$ Number of pivot positions.
4. The columns of a matrix A are linearly independent.

THEOREM: Let A be an $m \times n$ matrix. Then the following statements are logically equivalent:

1. A homogeneous system $A\mathbf{x} = \mathbf{0}$ has a nontrivial solution.
2. There are free variables.
3. Number of columns of $A >$ Number of pivot positions.
4. The columns of a matrix A are linearly dependent.
5. At least one column of A is a linear combination of other columns.

EXAMPLE: Let $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$ and $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ be the same as above.

We have

$$[\mathbf{u}_1\ \mathbf{u}_2\ \mathbf{u}_3] = \begin{bmatrix} 1 & -2 & 3 \\ 1 & -1 & 5 \\ 2 & -4 & 6 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

Since the number of pivots (two) is less than the number of columns (three), the vectors $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$ are linearly dependent. Similarly,

$$[\mathbf{v}_1\ \mathbf{v}_2\ \mathbf{v}_3] = \begin{bmatrix} 1 & -2 & 3 \\ 1 & -1 & 5 \\ 2 & -4 & 7 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$$

Since the number of pivots (three) is equal to the number of columns (three), the vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ are linearly independent.

THEOREM: The columns of an $n \times n$ matrix A are linearly independent if, and only if, $\det A \neq 0$.

EXAMPLE: Let $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$ and $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ be the same as above.

Since

$$\det([\mathbf{u}_1\ \mathbf{u}_2\ \mathbf{u}_3]) = \begin{vmatrix} 1 & -2 & 3 \\ 1 & -1 & 5 \\ 2 & -4 & 6 \end{vmatrix} = \begin{vmatrix} 1 & -2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{vmatrix} = 0$$

the vectors $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$ are linearly dependent. Similarly,

$$\det([\mathbf{v}_1\ \mathbf{v}_2\ \mathbf{v}_3]) = \begin{vmatrix} 1 & -2 & 3 \\ 1 & -1 & 5 \\ 2 & -4 & 7 \end{vmatrix} = \begin{vmatrix} 1 & -2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{vmatrix} = 1$$

Since the determinant is not equal to zero, the vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ are linearly independent.

EXAMPLE: Let

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \mathbf{v}_4 = \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix}$$

(a) The vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ are linearly independent, since the number of pivots in $[\mathbf{v}_1\ \mathbf{v}_2\ \mathbf{v}_3]$ (three) is equal to the number of columns (three).

These vectors span $\mathbb{R}^3$, since the number of pivots (three) is equal to the number of rows (three).

(b) The vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4$ are linearly dependent, since the number of pivots in $[\mathbf{v}_1\ \mathbf{v}_2\ \mathbf{v}_3\ \mathbf{v}_4]$ (three) is less than the number of columns (four).

These vectors span $\mathbb{R}^3$, since the number of pivots (three) is equal to the number of rows (three).

(c) The vectors $\mathbf{v}_1, \mathbf{v}_2$ are linearly independent, since the number of pivots in $[\mathbf{v}_1\ \mathbf{v}_2]$ (two) is equal to the number of columns (two).

These vectors do not span $\mathbb{R}^3$, since the number of pivots (two) is less than the number of rows (three).

(d) The vectors $\mathbf{v}_3, \mathbf{v}_4$ are linearly dependent, since the number of pivots in $[\mathbf{v}_3\ \mathbf{v}_4]$ (one) is less than the number of columns (two).

These vectors do not span $\mathbb{R}^3$, since the number of pivots (one) is less than the number of rows (three).

EXAMPLE: Let

$$A = \begin{bmatrix} 1 & -4 & -2 & 3 & -5 \\ 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & -4 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

(a) Are the columns of A linearly independent?

Solution: No. The number of columns (five) $>$ number of pivots (three).

(b) Are the 1st, 2nd, and 3rd columns of A linearly independent?

Solution: No. The number of columns (three) $>$ number of pivots (two).

(c) Are the 1st, 3rd, and 4th columns of A linearly independent?

Solution: Yes. The number of columns (three) $=$ number of pivots (three).

(d) Are the 1st and 2nd columns of A linearly independent?

Solution: No. The number of columns (two) $>$ number of pivots (one).

(e) Are the 4th and 5th columns of A linearly independent?

Solution: Yes. Indeed, we have

$$\begin{bmatrix} 3 & -5 \\ 0 & -1 \\ 1 & -4 \\ 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & -4 \\ 3 & -5 \\ 0 & -1 \\ 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & -4 \\ 0 & 7 \\ 0 & -1 \\ 0 & 0 \end{bmatrix}$$

Since the number of columns (two) $=$ number of pivots (two), the 4th and 5th columns of A are linearly independent.

EXAMPLE: The vectors

$$1+t^3, \quad 3+t-2t^2, \quad -t+3t^2-t^3$$

are linearly independent. Indeed,

$$\begin{bmatrix} 1 & 3 & 0 \\ 0 & 1 & -1 \\ 0 & -2 & 3 \\ 1 & 0 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 & 0 \\ 0 & 1 & -1 \\ 0 & -2 & 3 \\ 0 & -3 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \\ 0 & 0 & -4 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

Since there is a pivot in every column, the vectors are linearly independent.

EXAMPLE: The vectors

$$1-3t+5t^2, \ -3+5t-7t^2, \ -4+5t-6t^2, \ 1-t^2$$

are linearly dependent. Indeed,

$$\begin{bmatrix} 1 & -3 & -4 & 1 \\ -3 & 5 & 5 & 0 \\ 5 & -7 & -6 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & -3 & -4 & 1 \\ 0 & -4 & -7 & 3 \\ 0 & 8 & 14 & -6 \end{bmatrix} \sim \begin{bmatrix} 1 & -3 & -4 & 1 \\ 0 & 4 & 7 & -3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Since the number of pivots is less than the number of columns, the vectors are linearly dependent.

Alternate Characterizations of Linear Independence

THEOREM 4.8: Vectors $\mathbf{v}_1, \ldots, \mathbf{v}_p$ are linearly dependent if and only if at least one of these vectors is a linear combination of the others.

COROLLARY 1: A set S in a vector space V is linearly independent if and only if there is no vector $\mathbf{v} \in S$ such that $\mathbf{v} \in \operatorname{span}(S - \{\mathbf{v}\})$. A set S in a vector space V is linearly dependent if and only if there is some vector $\mathbf{v} \in S$ such that $\mathbf{v} \in \operatorname{span}(S - \{\mathbf{v}\})$.

COROLLARY 2: A nonempty set of vectors $S = \{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ is linearly independent if and only if
(1) $\mathbf{v}_1 \neq \mathbf{0}$ and (2) for each $k, 2 \leq k \leq n$, $\mathbf{v}_k \notin \operatorname{span}\{\mathbf{v}_1, \ldots, \mathbf{v}_{k-1}\}$.

Uniqueness of Expression of a Vector as a Linear Combination

THEOREM 4.9: Let S be a nonempty finite subset of a vector space V. Then S is linearly independent if and only if every vector $\mathbf{v} \in \operatorname{span}(S)$ can be expressed *uniquely* as a linear combination of the elements of S.

Proof: Let $S = \{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$.

Suppose first that S is linearly independent. Assume that $\mathbf{v} \in \operatorname{span}(S)$ can be expressed both as

$$\mathbf{v} = a_1\mathbf{v}_1 + \ldots + a_n\mathbf{v}_n \quad \text{and} \quad \mathbf{v} = b_1\mathbf{v}_1 + \ldots + b_n\mathbf{v}_n$$

then

$$\mathbf{0} = \mathbf{v} - \mathbf{v} = (a_1 - b_1)\mathbf{v}_1 + \ldots + (a_n - b_n)\mathbf{v}_n$$

Since S is a linearly independent set, each $a_i - b_i = 0$, by the definition of linear independence, and thus $a_i = b_i$ for all i.

Conversely, assume every vector in $\operatorname{span}(S)$ can be uniquely expressed as a linear combination of elements of S. Since $\mathbf{0} \in \operatorname{span}(S)$, there is exactly one linear combination $a_1\mathbf{v}_1 + \ldots + a_n\mathbf{v}_n$ of elements of S that equals $\mathbf{0}$. But the fact that $\mathbf{0} = 0\mathbf{v}_1 + \ldots + 0\mathbf{v}_n$ together with the uniqueness of expression for $\mathbf{0}$ means $a_1, \ldots, a_n$ are all zero. Thus, by the definition of linear independence, S is linearly independent. ■

Linear Independence of Infinite Sets

DEFINITION: An infinite subset S of a vector space V is **linearly dependent** if and only if there is some finite subset T of S such that T is linearly dependent. S is **linearly independent** if and only if S is not linearly dependent (that is, every finite subset T of S is linearly independent).

EXAMPLE: Consider the subset S of M_{22} consisting of all nonsingular 2×2 matrices. We will show that S is linearly dependent.

Let $T = \{I_2, 2I_2\}$, a subset of S. Clearly, since the second element of T is a scalar multiple of the first element of T, T is a linearly dependent set. Hence, S is linearly dependent, since one of its finite subsets is linearly dependent.

EXAMPLE: Let

$$S = \{1, 1 + x, 1 + x + x^2, 1 + x + x^2 + x^3, \dots\}$$

an infinite subset of P (the vector space of all polynomials). We will show that S is linearly independent.

Suppose $T = \{\mathbf{p}_1, \dots, \mathbf{p}_n\}$ is a finite subset of S, with the polynomials written in order of increasing degree. Also suppose that

$$a_1\mathbf{p}_1 + \dots + a_n\mathbf{p}_n = \mathbf{0}$$

We need to show that $a_1 = a_2 = \dots = a_n = 0$. We prove this by contradiction. Suppose at least one a_i is nonzero. Let a_k be the last nonzero coefficient in the series. Then,

$$a_1\mathbf{p}_1 + \dots + a_k\mathbf{p}_k = \mathbf{0}, \quad \text{with} \quad a_k \neq 0$$

Hence,

$$p_k = -\frac{a_1}{a_k}\mathbf{p}_1 - \frac{a_2}{a_k}\mathbf{p}_2 - \dots - \frac{a_{k-1}}{a_k}\mathbf{p}_{k-1}$$

Because all the degrees of the polynomials in T are different and they were listed in order of increasing degree, this equation expresses $\mathbf{p}_k$ as a linear combination of polynomials whose degrees are lower than that of $\mathbf{p}_k$. This can not happen, and so we get our desired contradiction. ■