

# Span

DEFINITION: The vector  $\mathbf{y}$  defined by

$$\mathbf{y} = c_1\mathbf{v}_1 + \dots + c_p\mathbf{v}_p$$

where  $\mathbf{v}_1, \dots, \mathbf{v}_p$  are vectors from a vector space  $V$  and  $c_1, \dots, c_p$  are scalars, is called a **linear combination** of vectors  $\mathbf{v}_1, \dots, \mathbf{v}_p$ .

EXAMPLES:

1. Let

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

then  $\mathbf{y} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$  is a linear combination of  $\mathbf{v}_1$  and  $\mathbf{v}_2$ , since

$$\begin{bmatrix} 3 \\ 2 \end{bmatrix} = 3 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + 2 \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

2. Let

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 4 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 2 \\ 7 \end{bmatrix}$$

then  $\mathbf{y} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$  is a linear combination of  $\mathbf{v}_1$  and  $\mathbf{v}_2$ , since

$$\begin{bmatrix} 3 \\ 1 \end{bmatrix} = (-19) \begin{bmatrix} 1 \\ 4 \end{bmatrix} + 11 \begin{bmatrix} 2 \\ 7 \end{bmatrix}$$

3. Let

$$\mathbf{v}_1 = \begin{bmatrix} 3 \\ -3 \\ 6 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 5 \\ -2 \\ 1 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} -4 \\ 4 \\ -8 \end{bmatrix}$$

then  $\mathbf{y} = \begin{bmatrix} 7 \\ -1 \\ -4 \end{bmatrix}$  is a linear combination of  $\mathbf{v}_1, \mathbf{v}_2$ , and  $\mathbf{v}_3$ . To find  $c_1, c_2$ , and  $c_3$  such that

$$\mathbf{y} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3$$

we apply the elementary row operations to the corresponding augmented matrix in order to come up with the reduced echelon form:

$$\begin{bmatrix} 3 & 5 & -4 & 7 \\ -3 & -2 & 4 & -1 \\ 6 & 1 & -8 & -4 \end{bmatrix} \sim \begin{bmatrix} 3 & 5 & -4 & 7 \\ 0 & 3 & 0 & 6 \\ 0 & -9 & 0 & -18 \end{bmatrix} \sim \begin{bmatrix} 3 & 5 & -4 & 7 \\ 0 & 3 & 0 & 6 \\ 0 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & \frac{5}{3} & -\frac{4}{3} & \frac{7}{3} \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix} \\ \sim \begin{bmatrix} 1 & 0 & -\frac{4}{3} & -1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

hence

$$\begin{cases} c_1 - \frac{4}{3}c_3 = -1 \\ c_2 = 2 \end{cases} \implies \begin{cases} c_1 = -1 + \frac{4}{3}c_3 \\ c_2 = 2 \\ c_3 \text{ is free} \end{cases}$$

For example, if  $c_3 = 3$ , then  $c_1 = 3$  and  $c_2 = 2$ . So,

$$\mathbf{y} = 3\mathbf{v}_1 + 2\mathbf{v}_2 + 3\mathbf{v}_3$$

that is,

$$\begin{bmatrix} 7 \\ -1 \\ -4 \end{bmatrix} = 3 \begin{bmatrix} 3 \\ -3 \\ 6 \end{bmatrix} + 2 \begin{bmatrix} 5 \\ -2 \\ 1 \end{bmatrix} + 3 \begin{bmatrix} -4 \\ 4 \\ -8 \end{bmatrix}$$

4. Let

$$\mathbf{v}_1 = 1 + t, \quad \mathbf{v}_2 = 1 + t^2, \quad \mathbf{v}_3 = t + t^2$$

then  $\mathbf{y} = -4 + 3t - 5t^2$  is a linear combination of  $\mathbf{v}_1, \mathbf{v}_2$ , and  $\mathbf{v}_3$ . To find  $c_1, c_2$ , and  $c_3$  such that

$$\mathbf{y} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3$$

we apply the elementary row operations to the corresponding augmented matrix in order to come up with the reduced echelon form:

$$\begin{aligned} \begin{bmatrix} 1 & 1 & 0 & -4 \\ 1 & 0 & 1 & 3 \\ 0 & 1 & 1 & -5 \end{bmatrix} &\sim \begin{bmatrix} 1 & 1 & 0 & -4 \\ 0 & -1 & 1 & 7 \\ 0 & 1 & 1 & -5 \end{bmatrix} \\ &\sim \begin{bmatrix} 1 & 1 & 0 & -4 \\ 0 & 1 & -1 & -7 \\ 0 & 1 & 1 & -5 \end{bmatrix} \\ &\sim \begin{bmatrix} 1 & 0 & 1 & 3 \\ 0 & 1 & -1 & -7 \\ 0 & 0 & 2 & 2 \end{bmatrix} \\ &\sim \begin{bmatrix} 1 & 0 & 1 & 3 \\ 0 & 1 & -1 & -7 \\ 0 & 0 & 1 & 1 \end{bmatrix} \\ &\sim \begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & -6 \\ 0 & 0 & 1 & 1 \end{bmatrix} \end{aligned}$$

hence  $c_1 = 2$ ,  $c_2 = -6$ , and  $c_3 = 1$ . So,

$$\mathbf{y} = 2\mathbf{v}_1 - 6\mathbf{v}_2 + \mathbf{v}_3$$

that is,

$$-4 + 3t - 5t^2 = 2(1 + t) - 6(1 + t^2) + t + t^2$$

DEFINITION: Let  $S$  be a nonempty subset of a vector space  $V$ . Then the **span** of  $S$  in  $V$  is the set of all possible (finite) linear combinations of the vectors in  $S$ . We use the notation  $\text{span}(S)$  to denote the span of  $S$  in  $V$ .

EXAMPLE: The vectors

$$\mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad \mathbf{e}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

span  $\mathbb{R}^2$ , since any vector

$$\mathbf{u} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

from  $\mathbb{R}^2$  can be written as a linear combination of  $\mathbf{e}_1, \mathbf{e}_2$ :

$$\mathbf{u} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 1 \cdot c_1 + 0 \cdot c_2 \\ 0 \cdot c_1 + 1 \cdot c_2 \end{bmatrix} = \begin{bmatrix} 1 \cdot c_1 \\ 0 \cdot c_1 \end{bmatrix} + \begin{bmatrix} 0 \cdot c_2 \\ 1 \cdot c_2 \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix} = c_1 \mathbf{e}_1 + c_2 \mathbf{e}_2$$

REMARK: In the same way one can show that the vectors

$$\mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{e}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix}, \quad \dots, \quad \mathbf{e}_n = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$$

span  $\mathbb{R}^n$ .

EXAMPLE: The vectors

$$1, \quad t, \quad t^2, \quad \dots, \quad t^n$$

span  $\mathbb{P}_n$  (the vector space of all polynomials of degree at most  $n$ ).

EXAMPLE: The vectors

$$M_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \quad M_2 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \end{bmatrix} \quad M_3 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \end{bmatrix}$$

$$M_4 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \quad M_5 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \quad M_6 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}$$

span the vector space of all  $3 \times 2$  matrices, since

$$\begin{aligned} \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix} &= \begin{bmatrix} a_{11} & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ a_{21} & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ a_{31} & 0 \end{bmatrix} + \begin{bmatrix} 0 & a_{12} \\ 0 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & a_{22} \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & a_{32} \end{bmatrix} \\ &= a_{11} \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} + a_{21} \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \end{bmatrix} + a_{31} \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \end{bmatrix} + a_{12} \begin{bmatrix} 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} + a_{22} \begin{bmatrix} 0 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} + a_{32} \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} \end{aligned}$$

EXAMPLE: Let

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 3 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 3 \\ 9 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} 4 \\ 7 \end{bmatrix}, \quad \text{and} \quad \mathbf{b} = \begin{bmatrix} 7 \\ 6 \end{bmatrix}$$

(a) Is  $\mathbf{b}$  in  $\text{span}(\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\})$ ?

Solution: We note that  $\mathbf{b}$  is in  $\text{span}(\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\})$  if and only if the system

$$\begin{cases} x_1 + 3x_2 + 4x_3 = 7 \\ 3x_1 + 9x_2 + 7x_3 = 6 \end{cases}$$

is consistent. We have

$$\begin{bmatrix} 1 & 3 & 4 & 7 \\ 3 & 9 & 7 & 6 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 & 4 & 7 \\ 0 & 0 & -5 & -15 \end{bmatrix}$$

Since the echelon form does not have a row of the form  $[0 \ 0 \ 0 \ c]$  with  $c \neq 0$ , it follows that the system above is consistent and therefore  $\mathbf{b}$  is in  $\text{span}(\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\})$ .

(b) Is  $\mathbf{b}$  in  $\text{span}(\{\mathbf{v}_1, \mathbf{v}_2\})$ ?

Solution: We note that  $\mathbf{b}$  is in  $\text{span}(\{\mathbf{v}_1, \mathbf{v}_2\})$  if and only if the system

$$\begin{cases} x_1 + 3x_2 = 7 \\ 3x_1 + 9x_2 = 6 \end{cases}$$

is consistent. We have

$$\begin{bmatrix} 1 & 3 & 7 \\ 3 & 9 & 6 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 & 7 \\ 0 & 0 & -15 \end{bmatrix}$$

Since the last row of the echelon form is  $[0 \ 0 \ -15]$ , it follows that the system above is inconsistent and therefore  $\mathbf{b}$  is not in  $\text{span}(\{\mathbf{v}_1, \mathbf{v}_2\})$ .

(c) Is  $\mathbf{b}$  in  $\text{span}(\{\mathbf{v}_1, \mathbf{v}_3\})$ ?

Solution: We note that  $\mathbf{b}$  is in  $\text{span}(\{\mathbf{v}_1, \mathbf{v}_3\})$  if and only if the system

$$\begin{cases} x_1 + 4x_2 = 7 \\ 3x_1 + 7x_2 = 6 \end{cases}$$

is consistent. We have

$$\begin{bmatrix} 1 & 4 & 7 \\ 3 & 7 & 6 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 & 7 \\ 0 & -5 & -15 \end{bmatrix}$$

Since the echelon form does not have a row of the form  $[0 \ 0 \ c]$  with  $c \neq 0$ , it follows that the system above is consistent and therefore  $\mathbf{b}$  is in  $\text{span}(\{\mathbf{v}_1, \mathbf{v}_3\})$ .

EXAMPLE: Let

$$\mathbf{v}_1 = 1 + t, \quad \mathbf{v}_2 = 1 + t^2, \quad \mathbf{v}_3 = t + t^2$$

then  $\mathbf{y} = -4 + 3t - 5t^2$  is in  $\text{span}(\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\})$  (see the corresponding example above).

THEOREM 4.5: Let  $S$  be a nonempty subset of a vector space  $V$ . Then:

- (1)  $S \subseteq \text{span}(S)$ .
- (2)  $\text{Span}(S)$  is a subspace of  $V$  (under the same operations as  $V$ ).
- (3) If  $W$  is a subspace of  $V$  with  $S \subseteq W$ , then  $\text{span}(S) \subseteq W$ .
- (4)  $\text{Span}(S)$  is the smallest subspace of  $V$  containing  $S$ .

EXAMPLE: The set  $W$  of all vectors

$$\mathbf{u} = \begin{bmatrix} x \\ 0 \\ z \end{bmatrix} \quad \text{where } x, z \text{ are all real numbers}$$

is a vector space. Indeed, since

$$\begin{bmatrix} x \\ 0 \\ z \end{bmatrix} = \begin{bmatrix} x \cdot 1 + z \cdot 0 \\ x \cdot 0 + z \cdot 0 \\ x \cdot 0 + z \cdot 1 \end{bmatrix} = x \underbrace{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}}_{\mathbf{v}_1} + z \underbrace{\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}}_{\mathbf{v}_2}$$

it follows that  $W = \text{span}(\{\mathbf{v}_1, \mathbf{v}_2\})$ . Therefore  $W$  is a subspace of  $\mathbb{R}^3$  by the Theorem above, part 2.

EXAMPLE: Let  $W$  be the set of all vectors of the form

$$\begin{bmatrix} a - b \\ b - c \\ c - a \\ b \end{bmatrix} \quad \text{where } a, b, c \text{ are all real numbers}$$

is a vector space. Indeed, since

$$\begin{aligned} \begin{bmatrix} a - b \\ b - c \\ c - a \\ b \end{bmatrix} &= \begin{bmatrix} 1 \cdot a + (-1) \cdot b + 0 \cdot c \\ 0 \cdot a + 1 \cdot b + (-1) \cdot c \\ (-1) \cdot a + 0 \cdot b + 1 \cdot c \\ 0 \cdot a + 1 \cdot b + 0 \cdot c \end{bmatrix} \\ &= a \underbrace{\begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \end{bmatrix}}_{\mathbf{v}_1} + b \underbrace{\begin{bmatrix} -1 \\ 1 \\ 0 \\ 1 \end{bmatrix}}_{\mathbf{v}_2} + c \underbrace{\begin{bmatrix} 0 \\ -1 \\ 1 \\ 0 \end{bmatrix}}_{\mathbf{v}_3} \end{aligned}$$

it follows that  $W = \text{span}(\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\})$ . Therefore  $W$  is a subspace of  $\mathbb{R}^4$  by the Theorem above, part 2.

COROLLARY 4.6: Let  $V$  be a vector space, and let  $S_1$  and  $S_2$  be subsets of  $V$  with  $S_1 \subseteq S_2$ . Then  $\text{span}(S_1) \subseteq \text{span}(S_2)$ .

**THEOREM:** Let  $\mathbf{v}_1, \dots, \mathbf{v}_p$  be vectors from  $\mathbb{R}^n$  and let  $B$  be an echelon form of the matrix  $A = [\mathbf{v}_1 \dots \mathbf{v}_p]$ . Then  $\mathbf{v}_1, \dots, \mathbf{v}_p$  span  $\mathbb{R}^n$  if and only if  $B$  has a pivot in every row (that is, the number of pivots is  $n$ ).

**EXAMPLE:** Let

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \quad \mathbf{v}_4 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

(a) The vectors  $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4$  span  $\mathbb{R}^3$ , since matrix

$$A = [\mathbf{v}_1 \ \mathbf{v}_2 \ \mathbf{v}_3 \ \mathbf{v}_4] = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

has a pivot in every row.

(b) The vectors  $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$  span  $\mathbb{R}^3$ , since matrix

$$A = [\mathbf{v}_1 \ \mathbf{v}_2 \ \mathbf{v}_3] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

has a pivot in every row.

(c) The vectors  $\mathbf{v}_1, \mathbf{v}_2$  do not span  $\mathbb{R}^3$ , since matrix

$$A = [\mathbf{v}_1 \ \mathbf{v}_2] = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

does not have a pivot in every row.

(d) The vectors  $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_4$  span  $\mathbb{R}^3$ , since matrix

$$A = [\mathbf{v}_1 \ \mathbf{v}_2 \ \mathbf{v}_4] = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 3 \end{bmatrix}$$

has a pivot in every row.

(e) The vectors  $\mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4$  span  $\mathbb{R}^3$ . Indeed, we have

$$A = [\mathbf{v}_2 \ \mathbf{v}_3 \ \mathbf{v}_4] = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 2 \\ 0 & 1 & 3 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix}$$

We see that the echelon form has a pivot in every row.

EXAMPLE: Let

$$\mathbf{v}_1 = \begin{bmatrix} 5 \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 7 \\ 2 \\ -6 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} 9 \\ 4 \\ -8 \end{bmatrix}, \quad \mathbf{v}_4 = \begin{bmatrix} 9 \\ 4 \\ -12 \end{bmatrix}$$

(a) The vectors  $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4$  span  $\mathbb{R}^3$ . Indeed, we have

$$A = [\mathbf{v}_1 \ \mathbf{v}_2 \ \mathbf{v}_3 \ \mathbf{v}_4] = \begin{bmatrix} 5 & 7 & 9 & 9 \\ 0 & 2 & 4 & 4 \\ 0 & -6 & -8 & -12 \end{bmatrix} \sim \begin{bmatrix} 5 & 7 & 9 & 9 \\ 0 & 2 & 4 & 4 \\ 0 & 0 & 4 & 0 \end{bmatrix}$$

We see that the echelon form has a pivot in every row.

(b) The vectors  $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$  span  $\mathbb{R}^3$ . Indeed, we have

$$A = [\mathbf{v}_1 \ \mathbf{v}_2 \ \mathbf{v}_3] = \begin{bmatrix} 5 & 7 & 9 \\ 0 & 2 & 4 \\ 0 & -6 & -8 \end{bmatrix} \sim \begin{bmatrix} 5 & 7 & 9 \\ 0 & 2 & 4 \\ 0 & 0 & 4 \end{bmatrix}$$

We see that the echelon form has a pivot in every row.

(c) The vectors  $\mathbf{v}_1, \mathbf{v}_2$  do not span  $\mathbb{R}^3$ . Indeed, we have

$$A = [\mathbf{v}_1 \ \mathbf{v}_2] = \begin{bmatrix} 5 & 7 \\ 0 & 2 \\ 0 & -6 \end{bmatrix} \sim \begin{bmatrix} 5 & 7 \\ 0 & 2 \\ 0 & 0 \end{bmatrix}$$

We see that the echelon form does not have a pivot in every row.

(d) The vectors  $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_4$  do not span  $\mathbb{R}^3$ . Indeed, we have

$$A = [\mathbf{v}_1 \ \mathbf{v}_2 \ \mathbf{v}_4] = \begin{bmatrix} 5 & 7 & 9 \\ 0 & 2 & 4 \\ 0 & -6 & -12 \end{bmatrix} \sim \begin{bmatrix} 5 & 7 & 9 \\ 0 & 2 & 4 \\ 0 & 0 & 0 \end{bmatrix}$$

We see that the echelon form does not have a pivot in every row.

EXAMPLE: The vectors (polynomials)

$$\mathbf{v}_1 = 1 + t, \quad \mathbf{v}_2 = 1 + t^2, \quad \mathbf{v}_3 = t + t^2$$

span  $\mathbb{P}^2$ . Indeed, we have

$$\begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & 2 \end{bmatrix}$$

We see that the echelon form has a pivot in every row.

EXAMPLE: The vectors

$$\mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{e}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix}, \quad \dots, \quad \mathbf{e}_n = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$$

span  $\mathbb{R}^n$ , since the  $n \times n$  matrix

$$A = [\mathbf{e}_1 \ \mathbf{e}_2 \ \dots \ \mathbf{e}_n] = \begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 1 \end{bmatrix}$$

has a pivot in every row.

EXAMPLE: The vectors (polynomials)

$$1, \quad t, \quad t^2, \quad \dots, \quad t^n$$

span  $\mathbb{P}_n$ , since the  $(n+1) \times (n+1)$  matrix

$$\begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 1 \end{bmatrix}$$

has a pivot in every row.

EXAMPLE: Let  $a_1, \dots, a_6$  be any nonzero numbers. The vectors (matrices)

$$M_1 = \begin{bmatrix} a_1 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \quad M_2 = \begin{bmatrix} 0 & 0 \\ a_2 & 0 \\ 0 & 0 \end{bmatrix} \quad M_3 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ a_3 & 0 \end{bmatrix}$$

$$M_4 = \begin{bmatrix} 0 & a_4 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \quad M_5 = \begin{bmatrix} 0 & 0 \\ 0 & a_5 \\ 0 & 0 \end{bmatrix} \quad M_6 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & a_6 \end{bmatrix}$$

span the vector space of all  $3 \times 2$  matrices, since the matrix

$$\begin{bmatrix} a_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & a_2 & 0 & 0 & 0 & 0 \\ 0 & 0 & a_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & a_4 & 0 & 0 \\ 0 & 0 & 0 & 0 & a_5 & 0 \\ 0 & 0 & 0 & 0 & 0 & a_6 \end{bmatrix}$$

has a pivot in every row.



COROLLARY: Let  $\mathbf{v}_1, \dots, \mathbf{v}_n$  be vectors from  $\mathbb{R}^n$ . Then  $\mathbf{v}_1, \dots, \mathbf{v}_n$  span  $\mathbb{R}^n$  if and only if

$$\det([\mathbf{v}_1 \ \dots \ \mathbf{v}_n]) \neq 0$$

EXAMPLE: The vectors

$$\mathbf{u}_1 = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}, \quad \mathbf{u}_2 = \begin{bmatrix} -2 \\ -1 \\ -4 \end{bmatrix}, \quad \mathbf{u}_3 = \begin{bmatrix} 3 \\ 5 \\ 6 \end{bmatrix}$$

do not span  $\mathbb{R}^3$ , since

$$\det([\mathbf{u}_1 \ \mathbf{u}_2 \ \mathbf{u}_3]) = \begin{vmatrix} 1 & -2 & 3 \\ 1 & -1 & 5 \\ 2 & -4 & 6 \end{vmatrix} = \begin{vmatrix} 1 & -2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{vmatrix} = 0$$

Similarly, the vectors

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} -2 \\ -1 \\ -4 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} 3 \\ 5 \\ 7 \end{bmatrix}$$

span  $\mathbb{R}^3$ , since

$$\det([\mathbf{v}_1 \ \mathbf{v}_2 \ \mathbf{v}_3]) = \begin{vmatrix} 1 & -2 & 3 \\ 1 & -1 & 5 \\ 2 & -4 & 7 \end{vmatrix} = \begin{vmatrix} 1 & -2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{vmatrix} = 1$$

EXAMPLE: Let  $\mathbf{v}_1 = 3 + 2t$ ,  $\mathbf{v}_2 = 1 - t^2$ ,  $\mathbf{v}_3 = 7 + 4t - t^2$ .

(a) Do the vectors  $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$  span  $\mathbb{P}^2$ ?

Solution: Since

$$\begin{vmatrix} 3 & 1 & 7 \\ 2 & 0 & 4 \\ 0 & -1 & -1 \end{vmatrix} = - \begin{vmatrix} 2 & 4 \\ 0 & -1 \end{vmatrix} - (-1) \begin{vmatrix} 3 & 7 \\ 2 & 4 \end{vmatrix} = -(-2 - 0) - (-1)(12 - 14) = 0$$

the vectors  $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$  do not span  $\mathbb{P}^2$ .

(b) Is the vector  $11 + 6t - 2t^2$  in  $\text{span}(\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\})$ ?

Solution: We have

$$\begin{bmatrix} 3 & 1 & 7 & 11 \\ 2 & 0 & 4 & 6 \\ 0 & -1 & -1 & -2 \end{bmatrix} \sim \begin{bmatrix} 2 & 0 & 4 & 6 \\ 3 & 1 & 7 & 11 \\ 0 & -1 & -1 & -2 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 2 & 3 \\ 3 & 1 & 7 & 11 \\ 0 & -1 & -1 & -2 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 2 & 3 \\ 0 & 1 & 1 & 2 \\ 0 & -1 & -1 & -2 \end{bmatrix} \\ \sim \begin{bmatrix} 1 & 0 & 2 & 3 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Since the echelon form does not have a row of the form  $[0 \ 0 \ 0 \ c]$  with  $c \neq 0$ , it follows that  $11 + 6t - 2t^2$  is in  $\text{span}(\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\})$ .

## Simplifying $\text{Span}(S)$ using Row Reduction

### Method for Simplifying $\text{Span}(S)$ Using Row Reduction (Simplified Span Method)

Suppose that  $S$  is a finite subset of  $\mathbb{R}^n$  containing  $k$  vectors, with  $k \geq 2$ .

**Step 1:** Form a  $k \times n$  matrix  $A$  by using the vectors in  $S$  as the rows of  $A$ . (Thus,  $\text{span}(S)$  is the row space of  $A$ ).

**Step 2:** Let  $C$  be the reduced row echelon form matrix for  $A$ .

**Step 3:** Then, a simplified form for  $\text{span}(S)$  is given by the set of all linear combinations of the nonzero rows of  $C$ .

EXAMPLE: Let  $S$  be the subset

$$\{[-3, 6, -1, 1, -7], [1, -2, 2, 3, -1], [2, -4, 5, 8, -4]\}$$

of  $\mathbb{R}^5$ . Since

$$\begin{aligned} \begin{bmatrix} -3 & 6 & -1 & 1 & -7 \\ 1 & -2 & 2 & 3 & -1 \\ 2 & -4 & 5 & 8 & -4 \end{bmatrix} &\sim \begin{bmatrix} 1 & -2 & 2 & 3 & -1 \\ -3 & 6 & -1 & 1 & -7 \\ 2 & -4 & 5 & 8 & -4 \end{bmatrix} \\ &\sim \begin{bmatrix} 1 & -2 & 2 & 3 & -1 \\ 0 & 0 & 5 & 10 & -10 \\ 0 & 0 & 1 & 2 & -2 \end{bmatrix} \\ &\sim \begin{bmatrix} 1 & -2 & 2 & 3 & -1 \\ 0 & 0 & 1 & 2 & -2 \\ 0 & 0 & 1 & 2 & -2 \end{bmatrix} \\ &\sim \begin{bmatrix} 1 & -2 & 2 & 3 & -1 \\ 0 & 0 & 1 & 2 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \\ &\sim \begin{bmatrix} 1 & -2 & 0 & -1 & 3 \\ 0 & 0 & 1 & 2 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \end{aligned}$$

it follows that

$$\text{span}(S) = \text{span}\{[1, -2, 0, -1, 3], [0, 0, 1, 2, -2]\}$$

## A Spanning Set for an Eigenspace

Let

$$A = \begin{bmatrix} 4 & -1 & 6 \\ 2 & 1 & 6 \\ 2 & -1 & 8 \end{bmatrix}$$

It was shown in Section 3.4 that

$$H_1 = \left\{ t_1 \begin{bmatrix} 1/2 \\ 1 \\ 0 \end{bmatrix} + t_2 \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix} : t_1, t_2 \in \mathbb{R} \right\}$$

is the eigenspace of  $A$  corresponding to  $\lambda = 2$  and

$$H_2 = \left\{ t \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} : t \in \mathbb{R} \right\}$$

is the eigenspace of  $A$  corresponding to  $\lambda = 9$ . Therefore

$$\left\{ \begin{bmatrix} 1/2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix} \right\}$$

is a spanning set for  $H_1$  and

$$\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}$$

is a spanning set for  $H_2$ .