Span

DEFINITION: The vector $\mathbf{y}$ defined by

$$\mathbf{y} = c_1\mathbf{v}_1 + \ldots + c_p\mathbf{v}_p$$

where $\mathbf{v}_1, \ldots, \mathbf{v}_p$ are vectors from a vector space V and $c_1, \ldots, c_p$ are scalars, is called a **linear combination** of vectors $\mathbf{v}_1, \ldots, \mathbf{v}_p$.

EXAMPLES:

1. Let

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

then $\mathbf{y} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$ is a linear combination of $\mathbf{v}_1$ and $\mathbf{v}_2$, since

$$\begin{bmatrix} 3 \\ 2 \end{bmatrix} = 3\begin{bmatrix} 1 \\ 0 \end{bmatrix} + 2\begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

2. Let

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 4 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 2 \\ 7 \end{bmatrix}$$

then $\mathbf{y} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$ is a linear combination of $\mathbf{v}_1$ and $\mathbf{v}_2$, since

$$\begin{bmatrix} 3 \\ 1 \end{bmatrix} = (-19)\begin{bmatrix} 1 \\ 4 \end{bmatrix} + 11\begin{bmatrix} 2 \\ 7 \end{bmatrix}$$

3. Let

$$\mathbf{v}_1 = \begin{bmatrix} 3 \\ -3 \\ 6 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 5 \\ -2 \\ 1 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} -4 \\ 4 \\ -8 \end{bmatrix}$$

then $\mathbf{y} = \begin{bmatrix} 7 \\ -1 \\ -4 \end{bmatrix}$ is a linear combination of $\mathbf{v}_1, \mathbf{v}_2$, and $\mathbf{v}_3$. To find c_1, c_2, and c_3 such that

$$\mathbf{y} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3$$

we apply the elementary row operations to the corresponding augmented matrix in order to come up with the reduced echelon form:

$$\begin{bmatrix} 3 & 5 & -4 & 7 \\ -3 & -2 & 4 & -1 \\ 6 & 1 & -8 & -4 \end{bmatrix} \sim \begin{bmatrix} 3 & 5 & -4 & 7 \\ 0 & 3 & 0 & 6 \\ 0 & -9 & 0 & -18 \end{bmatrix} \sim \begin{bmatrix} 3 & 5 & -4 & 7 \\ 0 & 3 & 0 & 6 \\ 0 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & \frac{5}{3} & -\frac{4}{3} & \frac{7}{3} \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 0 & -\frac{4}{3} & -1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

hence

$$\begin{cases} c_1 - \dfrac{4}{3}c_3 = -1 \\ \\ c_2 = 2 \end{cases} \quad \Longrightarrow \quad \begin{cases} c_1 = -1 + \dfrac{4}{3}c_3 \\ c_2 = 2 \\ c_3 \text{ is free} \end{cases}$$

For example, if $c_3 = 3$, then $c_1 = 3$ and $c_2 = 2$. So,

$$\mathbf{y} = 3\mathbf{v}_1 + 2\mathbf{v}_2 + 3\mathbf{v}_3$$

that is,

$$\begin{bmatrix} 7 \\ -1 \\ -4 \end{bmatrix} = 3\begin{bmatrix} 3 \\ -3 \\ 6 \end{bmatrix} + 2\begin{bmatrix} 5 \\ -2 \\ 1 \end{bmatrix} + 3\begin{bmatrix} -4 \\ 4 \\ -8 \end{bmatrix}$$

4. Let

$$\mathbf{v}_1 = 1 + t, \quad \mathbf{v}_2 = 1 + t^2, \quad \mathbf{v}_3 = t + t^2$$

then $\mathbf{y} = -4 + 3t - 5t^2$ is a linear combination of $\mathbf{v}_1, \mathbf{v}_2$, and $\mathbf{v}_3$. Consider the coefficients of each polynomial as the coordinates of a vector in $\mathbb{R}^3$. To find c_1, c_2, and c_3 such that

$$\mathbf{y} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3$$

we apply the elementary row operations to the corresponding augmented matrix in order to come up with the reduced echelon form:

$$\begin{bmatrix} 1 & 1 & 0 & -4 \\ 1 & 0 & 1 & 3 \\ 0 & 1 & 1 & -5 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 0 & -4 \\ 0 & -1 & 1 & 7 \\ 0 & 1 & 1 & -5 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 1 & 0 & -4 \\ 0 & 1 & -1 & -7 \\ 0 & 1 & 1 & -5 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 1 & 3 \\ 0 & 1 & -1 & -7 \\ 0 & 0 & 2 & 2 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 1 & 3 \\ 0 & 1 & -1 & -7 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & -6 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

hence $c_1 = 2$, $c_2 = -6$, and $c_3 = 1$. So,

$$\mathbf{y} = 2\mathbf{v}_1 - 6\mathbf{v}_2 + \mathbf{v}_3$$

that is,

$$-4 + 3t - 5t^2 = 2(1 + t) - 6(1 + t^2) + t + t^2$$

DEFINITION: Let S be a nonempty subset of a vector space V. Then the **span** of S in V is the set of all possible (finite) linear combinations of the vectors in S. We use the notation span(S) to denote the span of S in V.

EXAMPLE: The vectors

$$\mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad \mathbf{e}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

span $\mathbb{R}^2$, since any vector

$$\mathbf{u} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

from $\mathbb{R}^2$ can be written as a linear combination of $\mathbf{e}_1, \mathbf{e}_2$:

$$\mathbf{u} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 1 \cdot c_1 + 0 \cdot c_2 \\ 0 \cdot c_1 + 1 \cdot c_2 \end{bmatrix} = \begin{bmatrix} 1 \cdot c_1 \\ 0 \cdot c_1 \end{bmatrix} + \begin{bmatrix} 0 \cdot c_2 \\ 1 \cdot c_2 \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix} = c_1\mathbf{e}_1 + c_2\mathbf{e}_2$$

REMARK: In the same way one can show that the vectors

$$\mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{e}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix}, \quad \ldots, \quad \mathbf{e}_n = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$$

span $\mathbb{R}^n$.

EXAMPLE: The vectors

$$1, \quad t, \quad t^2, \quad \ldots, \quad t^n$$

span $\mathbb{P}_n$ (the vector space of all polynomials of degree at most n).

EXAMPLE: The vectors

$$M_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \qquad M_2 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \end{bmatrix} \qquad M_3 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \end{bmatrix}$$

$$M_4 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \qquad M_5 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \qquad M_6 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}$$

span the vector space of all 3×2 matrices, since

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix} = \begin{bmatrix} a_{11} & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ a_{21} & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ a_{31} & 0 \end{bmatrix} + \begin{bmatrix} 0 & a_{12} \\ 0 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & a_{22} \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & a_{32} \end{bmatrix}$$

$$= a_{11} \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} + a_{21} \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \end{bmatrix} + a_{31} \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \end{bmatrix} + a_{12} \begin{bmatrix} 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} + a_{22} \begin{bmatrix} 0 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} + a_{32} \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}$$

EXAMPLE: Let

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 3 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 3 \\ 9 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} 4 \\ 7 \end{bmatrix}, \quad \text{and} \quad \mathbf{b} = \begin{bmatrix} 7 \\ 6 \end{bmatrix}$$

(a) Is $\mathbf{b}$ in span$(\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\})$?

Solution: We note that $\mathbf{b}$ is in span$(\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\})$ if and only if the system

$$\begin{cases} x_1 + 3x_2 + 4x_3 = 7 \\ 3x_1 + 9x_2 + 7x_3 = 6 \end{cases}$$

is consistent. We have

$$\begin{bmatrix} 1 & 3 & 4 & 7 \\ 3 & 9 & 7 & 6 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 & 4 & 7 \\ 0 & 0 & -5 & -15 \end{bmatrix}$$

Since the echelon form does not have a row of the form $[0 \ \ 0 \ \ 0 \ \ c]$ with $c \neq 0$, it follows that the system above is consistent and therefore $\mathbf{b}$ is in span$(\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\})$.

(b) Is $\mathbf{b}$ in span$(\{\mathbf{v}_1, \mathbf{v}_2\})$?

Solution: We note that $\mathbf{b}$ is in span$(\{\mathbf{v}_1, \mathbf{v}_2\})$ if and only if the system

$$\begin{cases} x_1 + 3x_2 = 7 \\ 3x_1 + 9x_2 = 6 \end{cases}$$

is consistent. We have

$$\begin{bmatrix} 1 & 3 & 7 \\ 3 & 9 & 6 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 & 7 \\ 0 & 0 & -15 \end{bmatrix}$$

Since the last row of the echelon form is $[0 \ \ 0 \ \ -15]$, it follows that the system above is inconsistent and therefore $\mathbf{b}$ is not in span$(\{\mathbf{v}_1, \mathbf{v}_2\})$.

(c) Is $\mathbf{b}$ in span$(\{\mathbf{v}_1, \mathbf{v}_3\})$?

Solution: We note that $\mathbf{b}$ is in span$(\{\mathbf{v}_1, \mathbf{v}_3\})$ if and only if the system

$$\begin{cases} x_1 + 4x_2 = 7 \\ 3x_1 + 7x_2 = 6 \end{cases}$$

is consistent. We have

$$\begin{bmatrix} 1 & 4 & 7 \\ 3 & 7 & 6 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 & 7 \\ 0 & -5 & -15 \end{bmatrix}$$

Since the echelon form does not have a row of the form $[0 \ \ 0 \ \ c]$ with $c \neq 0$, it follows that the system above is consistent and therefore $\mathbf{b}$ is in span$(\{\mathbf{v}_1, \mathbf{v}_3\})$.

EXAMPLE: Let

$$\mathbf{v}_1 = 1 + t, \quad \mathbf{v}_2 = 1 + t^2, \quad \mathbf{v}_3 = t + t^2$$

then $\mathbf{y} = -4 + 3t - 5t^2$ is in span$(\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\})$ (see the corresponding example above).

THEOREM 4.5: Let S be a nonempty subset of a vector space V. Then:

(1) $S \subseteq \text{span}(S)$.

(2) Span(S) is a subspace of V (under the same operations as V).

(3) If W is a subspace of V with $S \subseteq W$, then span$(S) \subseteq W$.

(4) Span(S) is the smallest subspace of V containing S.

EXAMPLE: The set W of all vectors

$$\mathbf{u} = \begin{bmatrix} x \\ 0 \\ z \end{bmatrix} \quad \text{where} \quad x, z \quad \text{are all real numbers}$$

is a vector space. Indeed, since

$$\begin{bmatrix} x \\ 0 \\ z \end{bmatrix} = \begin{bmatrix} x \cdot 1 + z \cdot 0 \\ x \cdot 0 + z \cdot 0 \\ x \cdot 0 + z \cdot 1 \end{bmatrix} = x \underbrace{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}}_{\mathbf{v}_1} + z \underbrace{\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}}_{\mathbf{v}_2}$$

it follows that $W = \text{span}(\{\mathbf{v}_1, \mathbf{v}_2\})$. Therefore W is a subspace of $\mathbb{R}^3$ by the Theorem above, part 2.

EXAMPLE: Let W be the set of all vectors of the form

$$\begin{bmatrix} a-b \\ b-c \\ c-a \\ b \end{bmatrix} \quad \text{where} \quad a, b, c \quad \text{are all real numbers}$$

is a vector space. Indeed, since

$$\begin{bmatrix} a-b \\ b-c \\ c-a \\ b \end{bmatrix} = \begin{bmatrix} 1 \cdot a & + & (-1) \cdot b & + & 0 \cdot c \\ 0 \cdot a & + & 1 \cdot b & + & (-1) \cdot c \\ (-1) \cdot a & + & 0 \cdot b & + & 1 \cdot c \\ 0 \cdot a & + & 1 \cdot b & + & 0 \cdot c \end{bmatrix}$$

$$= a \underbrace{\begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \end{bmatrix}}_{\mathbf{v}_1} + b \underbrace{\begin{bmatrix} -1 \\ 1 \\ 0 \\ 1 \end{bmatrix}}_{\mathbf{v}_2} + c \underbrace{\begin{bmatrix} 0 \\ -1 \\ 1 \\ 0 \end{bmatrix}}_{\mathbf{v}_3}$$

it follows that $W = \text{span}(\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\})$. Therefore W is a subspace of $\mathbb{R}^4$ by the Theorem above, part 2.

COROLLARY 4.6: Let V be a vector space, and let S_1 and S_2 be subsets of V with $S_1 \subseteq S_2$. Then span$(S_1) \subseteq$ span(S_2).

THEOREM: Let $\mathbf{v}_1, \ldots, \mathbf{v}_p$ be vectors from $\mathbb{R}^n$ and let B be an echelon form of the matrix $A = [\mathbf{v}_1 \ldots \mathbf{v}_p]$. Then $\mathbf{v}_1, \ldots, \mathbf{v}_p$ span $\mathbb{R}^n$ if and only if B has a pivot in every row (that is, the number of pivots is n).

EXAMPLE: Let

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \quad \mathbf{v}_4 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

(a) The vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4$ span $\mathbb{R}^3$, since matrix

$$A = [\mathbf{v}_1\ \mathbf{v}_2\ \mathbf{v}_3\ \mathbf{v}_4] = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

has a pivot in every row.

(b) The vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ span $\mathbb{R}^3$, since matrix

$$A = [\mathbf{v}_1\ \mathbf{v}_2\ \mathbf{v}_3] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

has a pivot in every row.

(c) The vectors $\mathbf{v}_1, \mathbf{v}_2$ do not span $\mathbb{R}^3$, since matrix

$$A = [\mathbf{v}_1\ \mathbf{v}_2] = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

does not have a pivot in every row.

(d) The vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_4$ span $\mathbb{R}^3$, since matrix

$$A = [\mathbf{v}_1\ \mathbf{v}_2\ \mathbf{v}_4] = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 3 \end{bmatrix}$$

has a pivot in every row.

(e) The vectors $\mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4$ span $\mathbb{R}^3$. Indeed, we have

$$A = [\mathbf{v}_2\ \mathbf{v}_3\ \mathbf{v}_4] = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 2 \\ 0 & 1 & 3 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix}$$

We see that the echelon form has a pivot in every row.

EXAMPLE: Let

$$\mathbf{v}_1 = \begin{bmatrix} 5 \\ 0 \\ 0 \end{bmatrix}, \ \mathbf{v}_2 = \begin{bmatrix} 7 \\ 2 \\ -6 \end{bmatrix}, \ \mathbf{v}_3 = \begin{bmatrix} 9 \\ 4 \\ -8 \end{bmatrix}, \ \mathbf{v}_4 = \begin{bmatrix} 9 \\ 4 \\ -12 \end{bmatrix}$$

(a) The vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4$ span $\mathbb{R}^3$. Indeed, we have

$$A = [\mathbf{v}_1\ \mathbf{v}_2\ \mathbf{v}_3\ \mathbf{v}_4] = \begin{bmatrix} 5 & 7 & 9 & 9 \\ 0 & 2 & 4 & 4 \\ 0 & -6 & -8 & -12 \end{bmatrix} \sim \begin{bmatrix} 5 & 7 & 9 & 9 \\ 0 & 2 & 4 & 4 \\ 0 & 0 & 4 & 0 \end{bmatrix}$$

We see that the echelon form has a pivot in every row.

(b) The vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ span $\mathbb{R}^3$. Indeed, we have

$$A = [\mathbf{v}_1\ \mathbf{v}_2\ \mathbf{v}_3] = \begin{bmatrix} 5 & 7 & 9 \\ 0 & 2 & 4 \\ 0 & -6 & -8 \end{bmatrix} \sim \begin{bmatrix} 5 & 7 & 9 \\ 0 & 2 & 4 \\ 0 & 0 & 4 \end{bmatrix}$$

We see that the echelon form has a pivot in every row.

(c) The vectors $\mathbf{v}_1, \mathbf{v}_2$ do not span $\mathbb{R}^3$. Indeed, we have

$$A = [\mathbf{v}_1\ \mathbf{v}_2] = \begin{bmatrix} 5 & 7 \\ 0 & 2 \\ 0 & -6 \end{bmatrix} \sim \begin{bmatrix} 5 & 7 \\ 0 & 2 \\ 0 & 0 \end{bmatrix}$$

We see that the echelon form does not have a pivot in every row.

(d) The vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_4$ no not span $\mathbb{R}^3$. Indeed, we have

$$A = [\mathbf{v}_1\ \mathbf{v}_2\ \mathbf{v}_4] = \begin{bmatrix} 5 & 7 & 9 \\ 0 & 2 & 4 \\ 0 & -6 & -12 \end{bmatrix} \sim \begin{bmatrix} 5 & 7 & 9 \\ 0 & 2 & 4 \\ 0 & 0 & 0 \end{bmatrix}$$

We see that the echelon form does not have a pivot in every row.

EXAMPLE: The vectors (polynomials)

$$\mathbf{v}_1 = 1 + t, \quad \mathbf{v}_2 = 1 + t^2, \quad \mathbf{v}_3 = t + t^2$$

span $\mathbb{P}^2$. Indeed, we have

$$\begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & 2 \end{bmatrix}$$

We see that the echelon form has a pivot in every row.

EXAMPLE: The vectors

$$\mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{e}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix}, \quad \ldots, \quad \mathbf{e}_n = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$$

span $\mathbb{R}^n$, since the $n \times n$ matrix

$$A = [\mathbf{e}_1 \ \mathbf{e}_2 \ \ldots \ \mathbf{e}_n] = \begin{bmatrix} 1 & 0 & \ldots & 0 \\ 0 & 1 & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & \ldots & 1 \end{bmatrix}$$

has a pivot in every row.

EXAMPLE: The vectors (polynomials)

$$1, \quad t, \quad t^2, \quad \ldots, \quad t^n$$

span $\mathbb{P}_n$, since the $(n+1) \times (n+1)$ matrix

$$\begin{bmatrix} 1 & 0 & \ldots & 0 \\ 0 & 1 & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & \ldots & 1 \end{bmatrix}$$

has a pivot in every row.

EXAMPLE: Let $a_1, \ldots, a_6$ be any nonzero numbers. The vectors (matrices)

$$M_1 = \begin{bmatrix} a_1 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \qquad M_2 = \begin{bmatrix} 0 & 0 \\ a_2 & 0 \\ 0 & 0 \end{bmatrix} \qquad M_3 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ a_3 & 0 \end{bmatrix}$$

$$M_4 = \begin{bmatrix} 0 & a_4 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \qquad M_5 = \begin{bmatrix} 0 & 0 \\ 0 & a_5 \\ 0 & 0 \end{bmatrix} \qquad M_6 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & a_6 \end{bmatrix}$$

span the vector space of all 3×2 matrices, since the matrix

$$\begin{bmatrix} a_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & a_2 & 0 & 0 & 0 & 0 \\ 0 & 0 & a_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & a_4 & 0 & 0 \\ 0 & 0 & 0 & 0 & a_5 & 0 \\ 0 & 0 & 0 & 0 & 0 & a_6 \end{bmatrix}$$

has a pivot in every row.

COROLLARY: Let $\mathbf{v}_1, \ldots, \mathbf{v}_n$ be vectors from $\mathbb{R}^n$. Then $\mathbf{v}_1, \ldots, \mathbf{v}_n$ span $\mathbb{R}^n$ if and only if

$$\det([\mathbf{v}_1 \ \ldots \ \mathbf{v}_n]) \neq 0$$

EXAMPLE: The vectors

$$\mathbf{u}_1 = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}, \quad \mathbf{u}_2 = \begin{bmatrix} -2 \\ -1 \\ -4 \end{bmatrix}, \quad \mathbf{u}_3 = \begin{bmatrix} 3 \\ 5 \\ 6 \end{bmatrix}$$

do not span $\mathbb{R}^3$, since

$$\det([\mathbf{u}_1 \ \mathbf{u}_2 \ \mathbf{u}_3]) = \begin{vmatrix} 1 & -2 & 3 \\ 1 & -1 & 5 \\ 2 & -4 & 6 \end{vmatrix} = \begin{vmatrix} 1 & -2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{vmatrix} = 0$$

Similarly, the vectors

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} -2 \\ -1 \\ -4 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} 3 \\ 5 \\ 7 \end{bmatrix}$$

span $\mathbb{R}^3$, since

$$\det([\mathbf{v}_1 \ \mathbf{v}_2 \ \mathbf{v}_3]) = \begin{vmatrix} 1 & -2 & 3 \\ 1 & -1 & 5 \\ 2 & -4 & 7 \end{vmatrix} = \begin{vmatrix} 1 & -2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{vmatrix} = 1$$

EXAMPLE: Let $\mathbf{v}_1 = 3 + 2t,\ \mathbf{v}_2 = 1 - t^2,\ \mathbf{v}_3 = 7 + 4t - t^2$.

(a) Do the vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ span $\mathbb{P}^2$?

Solution: Consider the coefficients of each polynomial as the coordinates of a vector in $\mathbb{R}^3$. Since

$$\begin{vmatrix} 3 & 1 & 7 \\ 2 & 0 & 4 \\ 0 & -1 & -1 \end{vmatrix} = -\begin{vmatrix} 2 & 4 \\ 0 & -1 \end{vmatrix} - (-1)\begin{vmatrix} 3 & 7 \\ 2 & 4 \end{vmatrix} = -(-2-0) - (-1)(12-14) = 0$$

the vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ do not span $\mathbb{P}^2$.

(b) Is the vector $11 + 6t - 2t^2$ in $\operatorname{span}(\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\})$?

Solution: We have

$$\begin{bmatrix} 3 & 1 & 7 & 11 \\ 2 & 0 & 4 & 6 \\ 0 & -1 & -1 & -2 \end{bmatrix} \sim \begin{bmatrix} 2 & 0 & 4 & 6 \\ 3 & 1 & 7 & 11 \\ 0 & -1 & -1 & -2 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 2 & 3 \\ 3 & 1 & 7 & 11 \\ 0 & -1 & -1 & -2 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 2 & 3 \\ 0 & 1 & 1 & 2 \\ 0 & -1 & -1 & -2 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 2 & 3 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Since the echelon form does not have a row of the form $[0 \ \ 0 \ \ 0 \ \ c]$ with $c \neq 0$, it follows that $11 + 6t - 2t^2$ is in $\operatorname{span}(\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\})$.

Simplifying Span(S) using Row Reduction

Method for Simplifying Span(S) Using Row Reduction (Simplified Span Method)
Suppose that S is a finite subset of $\mathbb{R}^n$ containing k vectors, with $k \geq 2$.

Step 1: Form a $k \times n$ matrix A by using the vectors in S as the rows of A. (Thus, span(S) is the row space of A).

Step 2: Let C be the reduced row echelon form matrix for A.

Step 3: Then, a simplified form for span(S) is given by the set of all linear combinations of the nonzero rows of C.

EXAMPLE: Let S be the subset

$$\{[-3, 6, -1, 1, -7],\ [1, -2, 2, 3, -1],\ [2, -4, 5, 8, -4]\}$$

of $\mathbb{R}^5$. Since

$$\begin{bmatrix} -3 & 6 & -1 & 1 & -7 \\ 1 & -2 & 2 & 3 & -1 \\ 2 & -4 & 5 & 8 & -4 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 & 2 & 3 & -1 \\ -3 & 6 & -1 & 1 & -7 \\ 2 & -4 & 5 & 8 & -4 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & -2 & 2 & 3 & -1 \\ 0 & 0 & 5 & 10 & -10 \\ 0 & 0 & 1 & 2 & -2 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & -2 & 2 & 3 & -1 \\ 0 & 0 & 1 & 2 & -2 \\ 0 & 0 & 1 & 2 & -2 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & -2 & 2 & 3 & -1 \\ 0 & 0 & 1 & 2 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & -2 & 0 & -1 & 3 \\ 0 & 0 & 1 & 2 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

it follows that

$$\text{span}(S) = \text{span}\{[1, -2, 0, -1, 3],\ [0, 0, 1, 2, -2]\}$$

A Spanning Set for an Eigenspace

Let

$$A = \begin{bmatrix} 4 & -1 & 6 \\ 2 & 1 & 6 \\ 2 & -1 & 8 \end{bmatrix}$$

It was shown in Section 3.4 that

$$H_1 = \left\{ t_1 \begin{bmatrix} 1/2 \\ 1 \\ 0 \end{bmatrix} + t_2 \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix} \ :\ t_1, t_2 \in \mathbb{R} \right\}$$

is the eigenspace of A corresponding to $\lambda = 2$ and

$$H_2 = \left\{ t \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} : t \in \mathbb{R} \right\}$$

is the eigenspace of A corresponding to $\lambda = 9$. Therefore

$$\left\{ \begin{bmatrix} 1/2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix} \right\}$$

is a spanning set for H_1 and

$$\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}$$

is a spanning set for H_2.

Appendix

EXAMPLE: If V is the set of vectors

$$\begin{bmatrix} a+b \\ 0 \\ b \end{bmatrix}$$

in $\mathbb{R}^3$ where a and b are real numbers, then V is a vector space. Indeed, we have

$$\begin{bmatrix} a+b \\ 0 \\ b \end{bmatrix} = a \underbrace{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}}_{\mathbf{v}_1} + b \underbrace{\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}}_{\mathbf{v}_2}$$

Since $\mathbf{v}_1, \mathbf{v}_2$ are in the vector space $\mathbb{R}^3$ and V is the set of all linear combinations of $\mathbf{v}_1, \mathbf{v}_2$, it follows that V is a vector space by Theorem 4.5.

EXAMPLE: If H is the set of vectors

$$\begin{bmatrix} 4a-b \\ 2b \\ a-2b \\ a-b \end{bmatrix}$$

in $\mathbb{R}^4$ where a and b are real numbers, then H is a vector space. Indeed, we have

$$\begin{bmatrix} 4a-b \\ 2b \\ a-2b \\ a-b \end{bmatrix} = a \underbrace{\begin{bmatrix} 4 \\ 0 \\ 1 \\ 1 \end{bmatrix}}_{\mathbf{v}_1} + b \underbrace{\begin{bmatrix} -1 \\ 2 \\ -2 \\ -1 \end{bmatrix}}_{\mathbf{v}_2}$$

Since $\mathbf{v}_1, \mathbf{v}_2$ are in the vector space $\mathbb{R}^4$ and H is the set of all linear combinations of $\mathbf{v}_1, \mathbf{v}_2$, it follows that H is a vector space by Theorem 4.5.

EXAMPLE: If W is the set of all polynomials of the form $\mathbf{p}(t) = at^2 + bt$, where a, b are real numbers, then W is a vector space. Indeed, putting

$$\mathbf{v}_1 = t^2 \quad \text{and} \quad \mathbf{v}_2 = t$$

we get

$$\mathbf{p}(t) = a\mathbf{v}_1 + b\mathbf{v}_2$$

Since $\mathbf{v}_1, \mathbf{v}_2$ are in the vector space $\mathbb{P}_2$ and W is the set of all linear combinations of $\mathbf{v}_1, \mathbf{v}_2$, it follows that W is a vector space by Theorem 4.5.