Subspaces

DEFINITION: Let V be a vector space. Then W is a **subspace** of V if and only if W is a subset of V, and W is itself a vector space with the same operations as V.

THEOREM 4.2: Let V be a vector space, and let W be a nonempty subset of V using the same operations. Then W is a subspace of V if and only if W is closed under vector addition and scalar multiplication in V.

EXAMPLE: The set W of all vectors

$$\mathbf{u} = \begin{bmatrix} x \\ 0 \\ z \end{bmatrix} \quad \text{where} \quad x, z \quad \text{are all real numbers}$$

is a vector space, since W is a nonempty ($\mathbf{0}$ is in W) subset of the vector space $\mathbb{R}^3$ and it closed under vector addition and scalar multiplication:

1. W is closed under vector addition. Indeed, let

$$\mathbf{u}_1 = \begin{bmatrix} x_1 \\ 0 \\ z_1 \end{bmatrix} \quad \text{and} \quad \mathbf{u}_2 = \begin{bmatrix} x_2 \\ 0 \\ z_2 \end{bmatrix}$$

then

$$\mathbf{u}_1 + \mathbf{u}_2 = \begin{bmatrix} x_1 \\ 0 \\ z_1 \end{bmatrix} + \begin{bmatrix} x_2 \\ 0 \\ z_2 \end{bmatrix} = \begin{bmatrix} x_1 + x_2 \\ 0 \\ z_1 + z_2 \end{bmatrix} = \begin{bmatrix} x_3 \\ 0 \\ z_3 \end{bmatrix} \in W$$

where $x_3 = x_1 + x_2$ and $z_3 = z_1 + z_2$.

2. Similarly, W is closed under scalar multiplication, since

$$c\mathbf{u} = c \begin{bmatrix} x \\ 0 \\ z \end{bmatrix} = \begin{bmatrix} cx \\ 0 \\ cz \end{bmatrix} \in W$$

EXAMPLE: The set W of all vectors

$$\begin{bmatrix} x - y \\ y - z \\ z - x \\ y \end{bmatrix} \quad \text{where} \quad x, y, z \quad \text{are all real numbers}$$

is a vector space, since W is a nonempty ($\mathbf{0}$ is in W) subset of the vector space $\mathbb{R}^4$ and it closed under vector addition and scalar multiplication:

1. W is closed under vector addition. Indeed, let

$$\mathbf{u}_1 = \begin{bmatrix} x_1 - y_1 \\ y_1 - z_1 \\ z_1 - x_1 \\ y_1 \end{bmatrix} \quad \text{and} \quad \mathbf{u}_2 = \begin{bmatrix} x_2 - y_2 \\ y_2 - z_2 \\ z_2 - x_2 \\ y_2 \end{bmatrix}$$

then

$$\mathbf{u}_1 + \mathbf{u}_2 = \begin{bmatrix} x_1 - y_1 \\ y_1 - z_1 \\ z_1 - x_1 \\ y_1 \end{bmatrix} + \begin{bmatrix} x_2 - y_2 \\ y_2 - z_2 \\ z_2 - x_2 \\ y_2 \end{bmatrix} = \begin{bmatrix} (x_1 - y_1) + (x_2 - y_2) \\ (y_1 - z_1) + (y_2 - z_2) \\ (z_1 - x_1) + (z_2 - x_2) \\ y_1 + y_2 \end{bmatrix}$$

$$= \begin{bmatrix} (x_1 + x_2) - (y_1 + y_2) \\ (y_1 + y_2) - (z_1 + z_2) \\ (z_1 + z_2) - (x_1 + x_2) \\ y_1 + y_2 \end{bmatrix} = \begin{bmatrix} x_3 - y_3 \\ y_3 - z_3 \\ z_3 - x_3 \\ y_3 \end{bmatrix} \in W$$

where $x_3 = x_1 + x_2,\ y_3 = y_1 + y_2,\ z_3 = z_1 + z_2$.

2. Similarly, W is closed under scalar multiplication, since

$$c\mathbf{u} = c \begin{bmatrix} x - y \\ y - z \\ z - x \\ y \end{bmatrix} = \begin{bmatrix} cx - cy \\ cy - cz \\ cz - cx \\ cy \end{bmatrix} \in W$$

EXAMPLE: Let W be the set of all functions $x(t)$ which satisfy the differential equation

$$\frac{d^2x}{dt^2} - x = 0$$

with the sum of two functions and the product of a function by a number being defined in the usual manner. It is easy to verify that W is a vector space. We first note that W is a nonempty ($\mathbf{0}$ is in W) subset of the vector space of all real-valued functions. Let us show that it is closed under vector addition and scalar multiplication.

1. If $x_1(t)$ and $x_2(t)$ are in W, then $x_1(t) + x_2(t)$ is in W, since

$$\begin{aligned}(x_1 + x_2)'' - (x_1 + x_2) &= x_1'' + x_2'' - x_1 - x_2 \\ &= (x_1'' - x_1) + (x_2'' - x_2) \\ &= 0 + 0 \\ &= 0\end{aligned}$$

3. Similarly, if $x(t)$ is in W, then $cx(t)$ is in W, since

$$\begin{aligned}(cx)'' - (cx) &= cx'' - cx \\ &= c(x'' - x) \\ &= c \cdot 0 \\ &= 0\end{aligned}$$

DEFINITION: The vector $\mathbf{y}$ defined by

$$\mathbf{y} = c_1\mathbf{v}_1 + \ldots + c_n\mathbf{v}_n$$

where $\mathbf{v}_1, \ldots, \mathbf{v}_n$ are vectors from a vector space V and $c_1, \ldots, c_n$ are scalars, is called a **linear combination** of vectors $\mathbf{v}_1, \ldots, \mathbf{v}_n$.

THEOREM 4.3: Let W be a subspace of a vector space V, and let $\mathbf{v}_1, \ldots, \mathbf{v}_n$ be vectors in W. Then, for any scalars $c_1, \ldots, c_n$, we have $c_1\mathbf{v}_1 + \ldots + c_n\mathbf{v}_n \in W$.

Proof:

STEP 1: If $n = 1$, then we must show that if $\mathbf{v}_1 \in W$ and c_1 is a scalar, then $c_1\mathbf{v}_1 \in W$. But this is certainly true since the subspace W is closed under scalar multiplication.

STEP 2: Assume that the theorem is true for any linear combination of n vectors in W.

STEP 3: We must prove the theorem holds for a linear combination of $n+1$ vectors. Suppose $\mathbf{v}_1, \ldots, \mathbf{v}_n, \mathbf{v}_{n+1}$ are vectors in W, and $c_1, \ldots, c_n, c_{n+1}$ are scalars. We must show that $c_1\mathbf{v}_1 + \ldots + c_n\mathbf{v}_n + c_{n+1}\mathbf{v}_{n+1} \in W$. However, by the inductive hypothesis, we know that

$$c_1\mathbf{v}_1 + \ldots + c_n\mathbf{v}_n \in W$$

Also, $c_{n+1}\mathbf{v}_{n+1} \in W$, since W is closed under scalar multiplication. But since W is also closed under addition, the sum of any two vectors in W is again in W, so

$$(c_1\mathbf{v}_1 + \ldots + c_n\mathbf{v}_n) + (c_{n+1}\mathbf{v}_{n+1}) \in W \quad \blacksquare$$

THEOREM 4.4: Let A be an $n \times n$ matrix, and let λ be an eigenvalue for A, having eigenspace E_λ. Then E_λ is a subspace of $\mathbb{R}^n$.

Proof: Let λ be an eigenvalue for an $n \times n$ matrix A. By definition, the eigenspace E_λ of λ is the set of all n-vectors $\mathbf{x}$ having the property that

$$A\mathbf{x} = \lambda\mathbf{x}$$

including the zero n-vector. We will use Theorem 4.2 to show that E_λ is a subspace of $\mathbb{R}^n$.

Since $\mathbf{0} \in E_\lambda$, E_λ is a nonempty subset of $\mathbb{R}^n$. We must show that E_λ is closed under addition and scalar multiplication.

Let $\mathbf{x}_1, \mathbf{x}_2$ be any two vectors in E_λ. To show that $\mathbf{x}_1 + \mathbf{x}_2 \in E_\lambda$, we need to verify that

$$A(\mathbf{x}_1 + \mathbf{x}_2) = \lambda(\mathbf{x}_1 + \mathbf{x}_2)$$

But,

$$A(\mathbf{x}_1 + \mathbf{x}_2) = A\mathbf{x}_1 + A\mathbf{x}_2 = \lambda\mathbf{x}_1 + \lambda\mathbf{x}_2 = \lambda(\mathbf{x}_1 + \mathbf{x}_2)$$

therefore $\mathbf{x}_1 + \mathbf{x}_2 \in E_\lambda$. Similarly, let $\mathbf{x}$ be a vector in E_λ, and let c be a scalar. We must show that $c\mathbf{x} \in E_\lambda$. But,

$$A(c\mathbf{x}) = c(A\mathbf{x}) = c(\lambda\mathbf{x}) = \lambda(c\mathbf{x})$$

and so $c\mathbf{x} \in E_\lambda$. Hence, E_λ is a subspace of $\mathbb{R}^n$. $\blacksquare$