Introduction to Vector Spaces

DEFINITION: A **vector space** is a set V together with an operation called **vector addition** (a rule for adding two elements of V to obtain a third element of V) and another operation called **scalar multiplication** (a rule for multiplying a real number times an element of V to obtain a second element of V) on which the following ten properties hold: For every $\mathbf{u}, \mathbf{v}$, and $\mathbf{w}$ in V, and for every a and b in $\mathbb{R}$,

(A) The sum of $\mathbf{u}$ and $\mathbf{v}$, denoted by $\mathbf{u}+\mathbf{v}$, is in V.

(B) The scalar multiple of $\mathbf{u}$ by c, denoted by $c\mathbf{u}$, is in V.

(1) $\mathbf{u}+\mathbf{v}=\mathbf{v}+\mathbf{u}$.

(2) $(\mathbf{u}+\mathbf{v})+\mathbf{w}=\mathbf{u}+(\mathbf{v}+\mathbf{w})$.

(3) There is a unique element $\mathbf{0}$ in V, called the zero vector, such that $\mathbf{u}+\mathbf{0}=\mathbf{u}$.

(4) For each $\mathbf{u}$ in V, there is a unique vector $-\mathbf{u}$ in V such that $\mathbf{u}+(-\mathbf{u})=\mathbf{0}$.

(5) $a(\mathbf{u}+\mathbf{v})=a\mathbf{u}+a\mathbf{v}$.

(6) $(a+b)\mathbf{u}=a\mathbf{u}+b\mathbf{u}$.

(7) $a(b\mathbf{u})=(ab)\mathbf{u}$.

(8) $1\cdot\mathbf{u}=\mathbf{u}$.

The elements of a vector space V are called **vectors**.

EXAMPLE: Let $\mathbb{R}^n$ be the set of all $n\times 1$ matrices

$$\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

Define $\mathbf{u}+\mathbf{v}$ and $c\mathbf{u}$ as the vector addition and scalar multiplication, that is

$$\mathbf{u}+\mathbf{v}=\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}+\begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}=\begin{bmatrix} x_1+y_1 \\ x_2+y_2 \\ \vdots \\ x_n+y_n \end{bmatrix}$$

and

$$c\mathbf{u}=c\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}=\begin{bmatrix} cx_1 \\ cx_2 \\ \vdots \\ cx_n \end{bmatrix}$$

Then $\mathbb{R}^n$ is a vector space. In fact,

(A) $\mathbf{u}+\mathbf{v}$ is in $\mathbb{R}^n$.

(B) $c\mathbf{u}$ is in $\mathbb{R}^n$.

(1) Since $x_i + y_i = y_i + x_i$ for all real numbers x_i, y_i, it follows that $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$. Indeed,

$$\mathbf{u} + \mathbf{v} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} + \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} = \begin{bmatrix} x_1 + y_1 \\ x_2 + y_2 \\ \vdots \\ x_n + y_n \end{bmatrix}$$

$$= \begin{bmatrix} y_1 + x_1 \\ y_2 + x_2 \\ \vdots \\ y_n + x_n \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} + \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \mathbf{v} + \mathbf{u}$$

In the same way one can prove (2): Since $(x_i + y_i) + z_i = x_i + (y_i + z_i)$ for all real numbers x_i, y_i, z_i, it follows that $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$.

(3) The zero vector

$$\mathbf{0} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

satisfies $\mathbf{u} + \mathbf{0} = \mathbf{u}$, since

$$\mathbf{u} + \mathbf{0} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} = \begin{bmatrix} x_1 + 0 \\ x_2 + 0 \\ \vdots \\ x_n + 0 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \mathbf{u}$$

(4) For each $\mathbf{u} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$ in $\mathbb{R}^n$, there is the vector

$$-\mathbf{u} = \begin{bmatrix} -x_1 \\ -x_2 \\ \vdots \\ -x_n \end{bmatrix}$$

in $\mathbb{R}^n$ such that $\mathbf{u} + (-\mathbf{u}) = \mathbf{0}$, since

$$\mathbf{u} + (-\mathbf{u}) = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} + \begin{bmatrix} -x_1 \\ -x_2 \\ \vdots \\ -x_n \end{bmatrix} = \begin{bmatrix} x_1 + (-x_1) \\ x_2 + (-x_2) \\ \vdots \\ x_n + (-x_n) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} = \mathbf{0}$$

(5) $a(\mathbf{u}+\mathbf{v}) = a\mathbf{u} + a\mathbf{v}$, since

$$a(\mathbf{u}+\mathbf{v}) = a\left(\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} + \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}\right) = a\begin{bmatrix} x_1+y_1 \\ x_2+y_2 \\ \vdots \\ x_n+y_n \end{bmatrix}$$

$$= \begin{bmatrix} a(x_1+y_1) \\ a(x_2+y_2) \\ \vdots \\ a(x_n+y_n) \end{bmatrix}$$

$$= \begin{bmatrix} ax_1+ay_1 \\ ax_2+ay_2 \\ \vdots \\ ax_n+ay_n \end{bmatrix}$$

$$= \begin{bmatrix} ax_1 \\ ax_2 \\ \vdots \\ ax_n \end{bmatrix} + \begin{bmatrix} ay_1 \\ ay_2 \\ \vdots \\ ay_n \end{bmatrix} = a\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} + a\begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} = a\mathbf{u} + a\mathbf{v}$$

(6) $(a+b)\mathbf{u} = a\mathbf{u} + b\mathbf{u}$, since

$$(a+b)\mathbf{u} = (a+b)\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} (a+b)x_1 \\ (a+b)x_2 \\ \vdots \\ (a+b)x_n \end{bmatrix}$$

$$= \begin{bmatrix} ax_1+bx_1 \\ ax_2+bx_2 \\ \vdots \\ ax_n+bx_n \end{bmatrix}$$

$$= \begin{bmatrix} ax_1 \\ ax_2 \\ \vdots \\ ax_n \end{bmatrix} + \begin{bmatrix} bx_1 \\ bx_2 \\ \vdots \\ bx_n \end{bmatrix} = a\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} + b\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = a\mathbf{u} + b\mathbf{u}$$

(7) $a(b\mathbf{u}) = (ab)\mathbf{u}$, since

$$a(b\mathbf{u}) = a\left(b\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}\right) = a\begin{bmatrix} bx_1 \\ bx_2 \\ \vdots \\ bx_n \end{bmatrix} = \begin{bmatrix} a(bx_1) \\ a(bx_2) \\ \vdots \\ a(bx_n) \end{bmatrix} = \begin{bmatrix} (ab)x_1 \\ (ab)x_2 \\ \vdots \\ (ab)x_n \end{bmatrix} = (ab)\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = (ab)\mathbf{u}$$

(8) $1 \cdot \mathbf{u} = \mathbf{u}$, since

$$1 \cdot \mathbf{u} = 1 \cdot \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} 1 \cdot x_1 \\ 1 \cdot x_2 \\ \vdots \\ 1 \cdot x_n \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \mathbf{u}$$

EXAMPLE: Let V be the set of all $n \times m$ matrices

$$\begin{bmatrix} x_{11} & \dots & x_{1m} \\ x_{21} & \dots & x_{2m} \\ \dots & & \\ x_{n1} & \dots & x_{nm} \end{bmatrix}$$

Define $\mathbf{u} + \mathbf{v}$ and $c\mathbf{u}$ in the following way:

$$\mathbf{u} + \mathbf{v} = \begin{bmatrix} x_{11} & \dots & x_{1m} \\ x_{21} & \dots & x_{2m} \\ \dots & & \\ x_{n1} & \dots & x_{nm} \end{bmatrix} + \begin{bmatrix} y_{11} & \dots & y_{1m} \\ y_{21} & \dots & y_{2m} \\ \dots & & \\ y_{n1} & \dots & y_{nm} \end{bmatrix} = \begin{bmatrix} x_{11} + y_{11} & \dots & x_{1m} + y_{1m} \\ x_{21} + y_{21} & \dots & x_{2m} + y_{2m} \\ \dots & & \\ x_{n1} + y_{n1} & \dots & x_{nm} + y_{nm} \end{bmatrix}$$

and

$$a\mathbf{u} = a \begin{bmatrix} x_{11} & \dots & x_{1m} \\ x_{21} & \dots & x_{2m} \\ \dots & & \\ x_{n1} & \dots & x_{nm} \end{bmatrix} = \begin{bmatrix} ax_{11} & \dots & ax_{1m} \\ ax_{21} & \dots & ax_{2m} \\ \dots & & \\ ax_{n1} & \dots & ax_{nm} \end{bmatrix}$$

Then V is a vector space. In fact,
(A) $\mathbf{u} + \mathbf{v}$ is in V.

(B) $c\mathbf{u}$ is in V.

Properties (1) and (2) are true, since

$$x_{ij} + y_{ij} = y_{ij} + x_{ij}$$

and

$$(x_{ij} + y_{ij}) + z_{ij} = x_{ij} + (y_{ij} + z_{ij})$$

for all real numbers x_{ij}, y_{ij}, z_{ij}.

(3) The zero vector

$$\mathbf{0} = \begin{bmatrix} 0 & \dots & 0 \\ 0 & \dots & 0 \\ \dots & & \\ 0 & \dots & 0 \end{bmatrix}$$

satisfies $\mathbf{u} + \mathbf{0} = \mathbf{u}$.

(4) For each $\mathbf{u} = \begin{bmatrix} x_{11} & \dots & x_{1m} \\ x_{21} & \dots & x_{2m} \\ \dots & & \\ x_{n1} & \dots & x_{nm} \end{bmatrix}$ in V, there is the vector $-\mathbf{u} = \begin{bmatrix} -x_{11} & \dots & -x_{1m} \\ -x_{21} & \dots & -x_{2m} \\ \dots & & \\ -x_{n1} & \dots & -x_{nm} \end{bmatrix}$ in V such that $\mathbf{u} + (-\mathbf{u}) = \mathbf{0}$.

One can show that (5) – (8) are also true.

EXAMPLE: The set $\mathbb{P}_n$ of all polynomials of degree at most n:

$$\mathbf{p}(t) = a_n t^n + \ldots + a_2 t^2 + a_1 t + a_0$$

where the coefficients $a_n, \ldots, a_0$ and the variable t are real numbers is a vector space.

EXAMPLE: The set of all real-valued functions defined on $\mathbb{R}$ is a vector space.

EXAMPLE: The set $V = \{\mathbf{0}\}$ is a vector space with the rules for addition and multiplication given by $\mathbf{0} + \mathbf{0} = \mathbf{0}$ and $a\mathbf{0} = \mathbf{0}$ for every scalar (real number) a. Since $\mathbf{0}$ is the only possible result of either operation, V must be closed under both addition and scalar multiplication. A quick check verifies that the remaining eight properties also hold for V. This vector space is called the trivial vector space, and no smaller vector space is possible, because V is nonempty by definition.

EXAMPLE:
Let $\mathcal{V}$ be the set $\mathbb{R}^+$ of positive real numbers. This set is not a vector space under the usual operations of addition and scalar multiplication (why?). However, we can define new rules for these operations to make $\mathcal{V}$ a vector space. In what follows, we sometimes think of elements of $\mathbb{R}^+$ as abstract vectors (in which case we use boldface type, such as $\mathbf{v}$) or as the values on the positive real number line they represent (in which case we use italics, such as v).

To define "addition" on $\mathcal{V}$, we use *multiplication* of real numbers. That is,

$$\mathbf{v}_1 \oplus \mathbf{v}_2 = v_1 \cdot v_2$$

for every $\mathbf{v}_1$ and $\mathbf{v}_2$ in $\mathcal{V}$, where we use the symbol $\oplus$ for the "addition" operation on $\mathcal{V}$ to emphasize that this is not addition of real numbers. The definition of a vector space states only that vector addition must be a rule for combining two vectors to yield a third vector so that properties (1) through (8) hold. There is no stipulation that vector addition must be at all similar to ordinary addition of real numbers.[2]

We next define "scalar multiplication," $\odot$, on $\mathcal{V}$ by

$$a \odot \mathbf{v} = v^a$$

for every $a \in \mathbb{R}$ and $\mathbf{v} \in \mathcal{V}$.

From the given definitions, we see that if $\mathbf{v}_1$ and $\mathbf{v}_2$ are in $\mathcal{V}$ and a is in $\mathbb{R}$, then both $\mathbf{v}_1 \oplus \mathbf{v}_2$ and $a \odot \mathbf{v}_1$ are in $\mathcal{V}$, thus verifying the two closure properties. To prove the other eight properties, we assume that $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3 \in \mathcal{V}$ and that $a, b \in \mathbb{R}$. We then have the following:

Property (1): $\mathbf{v}_1 \oplus \mathbf{v}_2 = v_1 \cdot v_2 = v_2 \cdot v_1$ (by the commutative law of multiplication for real numbers) $= \mathbf{v}_2 \oplus \mathbf{v}_1$.

Property (2): $\mathbf{v}_1 \oplus (\mathbf{v}_2 \oplus \mathbf{v}_3) = \mathbf{v}_1 \oplus (v_2 \cdot v_3) = v_1 \cdot (v_2 \cdot v_3) = (v_1 \cdot v_2) \cdot v_3$ (by the associative law of multiplication for real numbers) $= (\mathbf{v}_1 \oplus \mathbf{v}_2) \cdot v_3 = (\mathbf{v}_1 \oplus \mathbf{v}_2) \oplus \mathbf{v}_3$.

Property (3): The number 1 in $\mathbb{R}^+$ acts as the zero vector $\mathbf{0}$ in $\mathcal{V}$ (why?).

Property (4): The additive inverse of $\mathbf{v}$ in $\mathcal{V}$ is the positive real number $(1/v)$, because $\mathbf{v} \oplus (1/v) = v \cdot (1/v) = 1$, the zero vector in $\mathcal{V}$.

Property (5): $a \odot (\mathbf{v}_1 \oplus \mathbf{v}_2) = a \odot (v_1 \cdot v_2) = (v_1 \cdot v_2)^a = v_1^a \cdot v_2^a = (a \odot \mathbf{v}_1) \cdot (a \odot \mathbf{v}_2) = (a \odot \mathbf{v}_1) \oplus (a \odot \mathbf{v}_2)$.

Property (6): $(a + b) \odot \mathbf{v} = v^{a+b} = v^a \cdot v^b = (a \odot \mathbf{v}) \cdot (b \odot \mathbf{v}) = (a \odot \mathbf{v}) \oplus (b \odot \mathbf{v})$.

Property (7): $(ab) \odot \mathbf{v} = v^{ab} = \left(v^b\right)^a = (b \odot \mathbf{v})^a = a \odot (b \odot \mathbf{v})$.

Property (8): $1 \odot \mathbf{v} = v^1 = \mathbf{v}$.

$\blacksquare$

Failure of the Vector Space Conditions

EXAMPLE: Here are some examples of sets that are *not* vector spaces:

1. The set of all vectors

$$\mathbf{u} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \quad \text{where } x_i \geq 0$$

is *not* a vector space, since property (B) fails.

2. The set of all vectors

$$\mathbf{u} = \begin{bmatrix} x \\ 1 \\ z \end{bmatrix} \quad \text{where } x, z \text{ are all real numbers}$$

is *not* a vector space (there is no $\mathbf{0}$). One can check, however, that the set V of all vectors

$$\mathbf{u} = \begin{bmatrix} x \\ 0 \\ z \end{bmatrix} \quad \text{where } x, z \text{ are all real numbers}$$

is a vector space (see Appendix I). Similarly, the set of all matrices

$$\mathbf{u} = \begin{bmatrix} 0 & x_1 & 0 \\ x_2 & x_3 & 0 \\ 0 & 0 & x_4 \\ x_5 & 0 & 0 \end{bmatrix} \quad \text{where } x_1, \ldots, x_5 \text{ are all real numbers}$$

is a vector space.

3. The set V of all vectors

$$\mathbf{u} = \begin{bmatrix} x \\ 0 \\ x^3 \end{bmatrix} \quad \text{where } x \text{ is any real number}$$

is *not* a vector space (property (A) fails). Indeed, if $\mathbf{u_1} =< 1, 0, 1 >$ and $\mathbf{u_2} =< 2, 0, 8 >$, then $\mathbf{u_1} + \mathbf{u_2} =< 3, 0, 9 >$ which is not in V.

4. The set of all polynomials of degree n with $n > 0$ is *not* a vector space (there is no $\mathbf{0}$). One can check, however, that the set of all polynomials of degree 0 *is* a vector space.

5. The set of all odd numbers is *not* a vector space (there is no $\mathbf{0}$). The set of all even numbers is *not* a vector space (property (B) fails).

6. The set of all positive numbers is *not* a vector space (there is no $\mathbf{0}$). The set of all nonnegative numbers is *not* a vector space (property (B) fails).

7. The set of all irrational numbers is *not* a vector space (there is no $\mathbf{0}$). The set of all rational numbers is *not* a vector space (property (B) fails, Number Theory requires).

Some Elementary Properties of Vector Spaces

> **Theorem 4.1** Let $\mathcal{V}$ be a vector space. Then, for every vector $\mathbf{v}$ in $\mathcal{V}$ and every real number a, we have
>
> | | |
> |---|---|
> | (1) $a\mathbf{0} = \mathbf{0}$ | Any scalar multiple of the zero vector yields the zero vector. |
> | (2) $0\mathbf{v} = \mathbf{0}$ | The scalar zero multiplied by any vector yields the zero vector. |
> | (3) $(-1)\mathbf{v} = -\mathbf{v}$ | The scalar -1 multiplied by any vector yields the additive inverse of that vector. |
> | (4) If $a\mathbf{v} = \mathbf{0}$, then $a = 0$ or $\mathbf{v} = \mathbf{0}$. | If a scalar multiplication yields the zero vector, then either the scalar is zero, or the vector is the zero vector, or both. |

Proof:

Part (1): By direct proof,

$$\begin{aligned}
a\mathbf{0} &= a\mathbf{0} + \mathbf{0} && \text{by property (3)}\\
&= a\mathbf{0} + (a\mathbf{0} + (-[a\mathbf{0}])) && \text{by property (4)}\\
&= (a\mathbf{0} + a\mathbf{0}) + (-[a\mathbf{0}]) && \text{by property (2)}\\
&= a(\mathbf{0} + \mathbf{0}) + (-[a\mathbf{0}]) && \text{by property (5)}\\
&= a\mathbf{0} + (-[a\mathbf{0}]) && \text{by property (3)}\\
&= \mathbf{0} && \text{by property (4)}
\end{aligned}$$

Part (2): By direct proof,

$$\begin{aligned}
0\mathbf{v} &= 0\mathbf{v} + \mathbf{0} && \text{by property (3)}\\
&= 0\mathbf{v} + (0\mathbf{v} + (-[0\mathbf{v}])) && \text{by property (4)}\\
&= (0\mathbf{v} + 0\mathbf{v}) + (-[0\mathbf{v}]) && \text{by property (2)}\\
&= (0 + 0)\mathbf{v} + (-[0\mathbf{v}]) && \text{by property (6)}\\
&= 0\mathbf{v} + (-[0\mathbf{v}]) && \text{by arithmetic}\\
&= \mathbf{0} && \text{by property (4)}
\end{aligned}$$

or

$$\begin{aligned}
0\mathbf{v} &= 0\mathbf{v} + \mathbf{0} && \text{by property (3)}\\
&= 0\mathbf{v} + (\mathbf{v} + (-\mathbf{v})) && \text{by property (4)}\\
&= (0\mathbf{v} + \mathbf{v}) + (-\mathbf{v}) && \text{by property (2)}\\
&= (0\mathbf{v} + 1\mathbf{v}) + (-\mathbf{v}) && \text{by property (8)}\\
&= (0 + 1)\mathbf{v} + (-\mathbf{v}) && \text{by property (6)}\\
&= 1\mathbf{v} + (-\mathbf{v}) && \text{by arithmetic}\\
&= \mathbf{v} + (-\mathbf{v}) && \text{by property (8)}\\
&= \mathbf{0} && \text{by property (4)}
\end{aligned}$$

REMARK: The following

$$0\mathbf{v} = (0-0)\mathbf{v} \overset{(1)}{=} 0\mathbf{v} - 0\mathbf{v} \overset{(2)}{=} \mathbf{0}$$

is only a short version of the proof above, since steps (1) and (2) should be justified.

Part (3): By direct proof,

$$\begin{aligned}
(-1)\mathbf{v} &= (-1)\mathbf{v} + \mathbf{0} && \text{by property (3)} \\
&= (-1)\mathbf{v} + (\mathbf{v} + (-\mathbf{v})) && \text{by property (4)} \\
&= ((-1)\mathbf{v} + \mathbf{v}) + (-\mathbf{v}) && \text{by property (2)} \\
&= ((-1)\mathbf{v} + 1\mathbf{v}) + (-\mathbf{v}) && \text{by property (8)} \\
&= ((-1) + 1)\mathbf{v} + (-\mathbf{v}) && \text{by property (6)} \\
&= 0\mathbf{v} + (-\mathbf{v}) && \text{by arithmetic} \\
&= \mathbf{0} + (-\mathbf{v}) && \text{by part (2) above} \\
&= -\mathbf{v} && \text{by property (3)}
\end{aligned}$$

Here is another way to prove it. First, note that

$$\begin{aligned}
\mathbf{v} + (-1)\mathbf{v} &= 1\mathbf{v} + (-1)\mathbf{v} && \text{by property (8)} \\
&= (1 + (-1))\mathbf{v} && \text{by property (6)} \\
&= 0\mathbf{v} && \text{by arithmetic} \\
&= 0 && \text{by part (2) above}
\end{aligned}$$

Therefore, $(-1)\mathbf{v}$ acts as *an* additive inverse for $\mathbf{v}$. We will finish the proof by showing that the additive inverse for $\mathbf{v}$ is unique. Hence, $(-1)\mathbf{v}$ will be *the* additive inverse of $\mathbf{v}$.

Suppose that $\mathbf{x}$ and $\mathbf{y}$ are both additive inverses for $\mathbf{v}$. Thus, $\mathbf{x}+\mathbf{v} = \mathbf{0}$ and $\mathbf{v}+\mathbf{y} = \mathbf{0}$. Hence,

$$\mathbf{x} = \mathbf{x} + \mathbf{0} = \mathbf{x} + (\mathbf{v} + \mathbf{y}) = (\mathbf{x} + \mathbf{v}) + \mathbf{y} = \mathbf{0} + \mathbf{y} = \mathbf{y}$$

Therefore, any two additive inverses of $\mathbf{v}$ are equal.

Part (4): This is an "If A then B or C" statement. Therefore, we assume that $a\mathbf{v} = \mathbf{0}$ and $a \neq 0$ and show that $\mathbf{v} = \mathbf{0}$. Now,

$$\begin{aligned}
\mathbf{v} &= 1\mathbf{v} && \text{by property (8)} \\
&= \left(\frac{1}{a} \cdot a\right)\mathbf{v} && \text{because } a \neq 0 \\
&= \left(\frac{1}{a}\right)(a\mathbf{v}) && \text{by property (7)} \\
&= \left(\frac{1}{a}\right)\mathbf{0} && \text{because } a\mathbf{v} = \mathbf{0} \\
&= \mathbf{0} && \text{by part (1) above}
\end{aligned}$$

Appendix I

Here we show that the set H of all vectors $< x, 0, z >$, where x, z are real numbers, is a vector space. Let

$$\mathbf{u} = \begin{bmatrix} x_1 \\ 0 \\ z_1 \end{bmatrix} \qquad \mathbf{v} = \begin{bmatrix} x_2 \\ 0 \\ z_2 \end{bmatrix} \qquad \mathbf{w} = \begin{bmatrix} x_3 \\ 0 \\ z_3 \end{bmatrix}$$

(A) $\mathbf{u} + \mathbf{v}$ is in H, since

$$\mathbf{u} + \mathbf{v} = \begin{bmatrix} x_1 \\ 0 \\ z_1 \end{bmatrix} + \begin{bmatrix} x_2 \\ 0 \\ z_2 \end{bmatrix} = \begin{bmatrix} x_1 + x_2 \\ 0 \\ z_1 + z_2 \end{bmatrix}$$

(B) $c\mathbf{u}$ is in H, since

$$c\mathbf{u} = c \begin{bmatrix} x_1 \\ 0 \\ z_1 \end{bmatrix} = \begin{bmatrix} cx_1 \\ 0 \\ cz_1 \end{bmatrix}$$

(1) We have

$$\mathbf{u} + \mathbf{v} = \begin{bmatrix} x_1 \\ 0 \\ z_1 \end{bmatrix} + \begin{bmatrix} x_2 \\ 0 \\ z_2 \end{bmatrix} = \begin{bmatrix} x_1 + x_2 \\ 0 \\ z_1 + z_2 \end{bmatrix} = \begin{bmatrix} x_2 + x_1 \\ 0 \\ z_2 + z_1 \end{bmatrix} = \begin{bmatrix} x_2 \\ 0 \\ z_2 \end{bmatrix} + \begin{bmatrix} x_1 \\ 0 \\ z_1 \end{bmatrix} = \mathbf{v} + \mathbf{u}$$

In the same way one can prove (2): Since $(x_1 + x_2) + x_3 = x_1 + (x_2 + x_3)$ and $(z_1 + z_2) + z_3 = z_1 + (z_2 + z_3)$, it follows that $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$.

(3) The zero vector

$$\mathbf{0} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

satisfies $\mathbf{u} + \mathbf{0} = \mathbf{u}$, since

$$\mathbf{u} + \mathbf{0} = \begin{bmatrix} x_1 \\ 0 \\ z_1 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} x_1 + 0 \\ 0 + 0 \\ z_1 + 0 \end{bmatrix} = \begin{bmatrix} x_1 \\ 0 \\ z_1 \end{bmatrix} = \mathbf{u}$$

(4) For each $\mathbf{u} = \begin{bmatrix} x_1 \\ 0 \\ z_1 \end{bmatrix}$ in H, there is the vector $-\mathbf{u} = \begin{bmatrix} -x_1 \\ 0 \\ -z_1 \end{bmatrix}$ in H such that $\mathbf{u} + (-\mathbf{u}) = \mathbf{0}$, since

$$\mathbf{u} + (-\mathbf{u}) = \begin{bmatrix} x_1 \\ 0 \\ z_1 \end{bmatrix} + \begin{bmatrix} -x_1 \\ 0 \\ -z_1 \end{bmatrix} = \begin{bmatrix} x_1 + (-x_1) \\ 0 + 0 \\ z_1 + (-z_1) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \mathbf{0}$$

(5) $a(\mathbf{u}+\mathbf{v}) = a\mathbf{u} + a\mathbf{v}$, since

$$\begin{aligned} a(\mathbf{u}+\mathbf{v}) = a\left(\begin{bmatrix} x_1 \\ 0 \\ z_1 \end{bmatrix} + \begin{bmatrix} x_2 \\ 0 \\ z_2 \end{bmatrix}\right) &= a\begin{bmatrix} x_1+x_2 \\ 0 \\ z_1+z_2 \end{bmatrix} \\ &= \begin{bmatrix} a(x_1+x_2) \\ 0 \\ a(z_1+z_2) \end{bmatrix} \\ &= \begin{bmatrix} ax_1+ax_2 \\ 0 \\ az_1+az_2 \end{bmatrix} \\ &= \begin{bmatrix} ax_1 \\ 0 \\ az_1 \end{bmatrix} + \begin{bmatrix} ax_2 \\ 0 \\ az_2 \end{bmatrix} \\ &= a\begin{bmatrix} x_1 \\ 0 \\ z_1 \end{bmatrix} + a\begin{bmatrix} x_2 \\ 0 \\ z_2 \end{bmatrix} = a\mathbf{u} + a\mathbf{v} \end{aligned}$$

(6) $(a+b)\mathbf{u} = a\mathbf{u} + b\mathbf{u}$, since

$$\begin{aligned} (a+b)\mathbf{u} = (a+b)\begin{bmatrix} x_1 \\ 0 \\ z_1 \end{bmatrix} &= \begin{bmatrix} (a+b)x_1 \\ 0 \\ (a+b)z_1 \end{bmatrix} \\ &= \begin{bmatrix} ax_1+bx_1 \\ 0 \\ az_1+bz_1 \end{bmatrix} \\ &= \begin{bmatrix} ax_1 \\ 0 \\ az_1 \end{bmatrix} + \begin{bmatrix} bx_1 \\ 0 \\ bz_1 \end{bmatrix} \\ &= a\begin{bmatrix} x_1 \\ 0 \\ z_1 \end{bmatrix} + b\begin{bmatrix} x_1 \\ 0 \\ z_1 \end{bmatrix} = a\mathbf{u} + b\mathbf{u} \end{aligned}$$

(7) $a(b\mathbf{u}) = (ab)\mathbf{u}$, since

$$a(b\mathbf{u}) = a\left(b\begin{bmatrix} x_1 \\ 0 \\ z_1 \end{bmatrix}\right) = a\begin{bmatrix} bx_1 \\ 0 \\ bz_1 \end{bmatrix} = \begin{bmatrix} a(bx_1) \\ 0 \\ a(bz_1) \end{bmatrix} = \begin{bmatrix} (ab)x_1 \\ 0 \\ (ab)z_1 \end{bmatrix} = (ab)\begin{bmatrix} x_1 \\ 0 \\ z_1 \end{bmatrix} = (ab)\mathbf{u}$$

(8) $1 \cdot \mathbf{u} = \mathbf{u}$, since

$$1 \cdot \mathbf{u} = 1 \cdot \begin{bmatrix} x_1 \\ 0 \\ z_1 \end{bmatrix} = \begin{bmatrix} 1 \cdot x_1 \\ 1 \cdot 0 \\ 1 \cdot z_1 \end{bmatrix} = \begin{bmatrix} x_1 \\ 0 \\ z_1 \end{bmatrix} = \mathbf{u}$$

Appendix II

EXAMPLE: Let H be the set of all vectors of the form

$$\begin{bmatrix} 3a+b \\ 4 \\ a-5b \end{bmatrix}$$

where a and b are arbitrary scalars. Show that H is not a vector space.

Solution: H is not a vector space, since $\mathbf{0} \notin H$ (the second entry is always nonzero).

EXAMPLE: Let V be the set of vectors

$$\begin{bmatrix} 2a \\ ab \\ 3b \end{bmatrix}$$

in $\mathbb{R}^3$ where a and b are real numbers. Then

(A) V is a vector space, since $\mathbf{0}$ is in V

(B) V is not a vector space, since $\mathbf{0}$ is not in V

(C) V is a vector space, since $\mathbf{u}+\mathbf{v}$ is from V for any $\mathbf{u}$ and $\mathbf{v}$ from V

(D) V is a vector space, since $c\mathbf{u}$ is from V for any $\mathbf{u}$ from V and any c from $\mathbb{R}$

(E) V is not a vector space, since there exist $\mathbf{u}$ and $\mathbf{v}$ from V such that $\mathbf{u}+\mathbf{v}$ is not from V

(F) None of the above

Solution: Let

$$\mathbf{u} = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix} \quad \text{(here } a=1, b=1\text{)}$$

and

$$\mathbf{v} = \begin{bmatrix} 4 \\ 4 \\ 6 \end{bmatrix} \quad \text{(here } a=2, b=2\text{)}$$

then

$$\mathbf{u}+\mathbf{v} = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix} + \begin{bmatrix} 4 \\ 4 \\ 6 \end{bmatrix} = \begin{bmatrix} 2+4 \\ 1+4 \\ 3+6 \end{bmatrix} = \begin{bmatrix} 6 \\ 5 \\ 9 \end{bmatrix}$$

However,

$$\begin{bmatrix} 6 \\ 5 \\ 9 \end{bmatrix} \neq \begin{bmatrix} 2a \\ ab \\ 3b \end{bmatrix}$$

since $6 = 2a$ implies $a = 3$, $9 = 3b$ implies $b = 3$, therefore $5 \neq ab$. So, the correct answer is E.

Info: This problem was given in Spring 2017 (Differential Equations and Linear Algebra, Midterm II). The average in the class for this problem was 60.2%.

EXAMPLE: Let V be the set of vectors

$$\begin{bmatrix} ab \\ a \\ b \end{bmatrix}$$

in $\mathbb{R}^3$ where a and b are real numbers. Show that V is not a vector space.

Solution: Let

$$\mathbf{u} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \qquad (\text{here } a = 1, b = 1)$$

and

$$\mathbf{v} = \begin{bmatrix} 4 \\ 2 \\ 2 \end{bmatrix} \qquad (\text{here } a = 2, b = 2)$$

then

$$\mathbf{u} + \mathbf{v} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 4 \\ 2 \\ 2 \end{bmatrix} = \begin{bmatrix} 1+4 \\ 1+2 \\ 1+2 \end{bmatrix} = \begin{bmatrix} 5 \\ 3 \\ 3 \end{bmatrix}$$

However,

$$\begin{bmatrix} 5 \\ 3 \\ 3 \end{bmatrix} \neq \begin{bmatrix} ab \\ a \\ b \end{bmatrix}$$

since otherwise $a = 3$, $b = 3$ and $ab = 5$, which is impossible.

EXAMPLE: Let V be the set of vectors

$$\begin{bmatrix} ab \\ 0 \\ b \end{bmatrix}$$

in $\mathbb{R}^3$ where a and b are real numbers. Show that V is not a vector space.

Solution: Let

$$\mathbf{u} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \quad \text{(here } a = 1, b = 1\text{)}$$

and

$$\mathbf{v} = \begin{bmatrix} -2 \\ 0 \\ -1 \end{bmatrix} \quad \text{(here } a = 2, b = -1\text{)}$$

then

$$\mathbf{u} + \mathbf{v} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + \begin{bmatrix} -2 \\ 0 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 + (-2) \\ 0 + 0 \\ 1 + (-1) \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix}$$

However,

$$\begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix} \neq \begin{bmatrix} ab \\ 0 \\ b \end{bmatrix}$$

since otherwise $b = 0$ and $ab = -1$, which is impossible.

REMARK 1: Note that if we set $a = 1, b = 1$ and $a = 2, b = 2$ for $\mathbf{u}$ and $\mathbf{v}$ (like we did in the two previous examples), we will *not* get a contradiction. Indeed, in this case we have

$$\mathbf{u} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \quad \text{and} \quad \mathbf{v} = \begin{bmatrix} 4 \\ 0 \\ 2 \end{bmatrix}$$

But

$$\mathbf{u} + \mathbf{v} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + \begin{bmatrix} 4 \\ 0 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 + 4 \\ 0 + 0 \\ 1 + 2 \end{bmatrix} = \begin{bmatrix} 5 \\ 0 \\ 3 \end{bmatrix}$$

which is equal to $\begin{bmatrix} ab \\ 0 \\ b \end{bmatrix}$ if $a = 5/3$ and $b = 3$.

REMARK 2: If V is the set of vectors

$$\begin{bmatrix} a + b \\ 0 \\ b \end{bmatrix}$$

in $\mathbb{R}^3$ where a and b are real numbers, then V *is* a vector space.

EXAMPLE: Let W_1 be the set of vectors

$$\mathbf{u} = \begin{bmatrix} x \\ x+1 \\ y-1 \end{bmatrix}$$

and let W_2 be the set of vectors

$$\mathbf{v} = \begin{bmatrix} x \\ y^2 \\ x+y \end{bmatrix}$$

where x and y are all real numbers. Then

(A) W_1 is a vector space, but W_2 is not a vector space.

(B) W_1 is not a vector space, but W_2 is a vector space.

(C) W_1 is a vector space and W_2 is a vector space.

(D) W_1 is a not vector space and W_2 is not a vector space. $\longleftarrow$ correct

Info: This problem was given in Fall 2017 (Differential Equations and Linear Algebra, Midterm Exam II). The average in the class for this problem was 71.4%.

EXAMPLE: Let W be the set of all polynomials of the form $\mathbf{p}(t) = at^2 + bt + 1$, where a, b are real numbers. Then

(A) W is not a vector space, since it is not closed under multiplication by a scalar.

(B) W is not a vector space, since $\mathbf{0}$ is not in W.

(C) W is not a vector space, since it is not closed under addition.

(D) All of the above $\longleftarrow$ correct

(E) None of the above

Info: This problem was given in Fall 2017 (Differential Equations and Linear Algebra, Final Exam II). The average in the class for this problem was 43.2%.

REMARK: If W is the set of all polynomials of the form $\mathbf{p}(t) = at^2 + bt$, where a, b are real numbers, then W *is* a vector space.

EXAMPLE: Let W be the set of singular 2×2 matrices under the usual operations. Then

(A) W is not a vector space, since it is not closed under addition. $\longleftarrow$ correct

(B) W is not a vector space, since $\mathbf{0}$ is not in W.

(C) W is not a vector space, since it is not closed under multiplication by a scalar.

(D) W is not a vector space, since it is not a subspace of the vector space of 2×2 matrices.

(E) None of the above

Info: This problem was given in Fall 2017 (Differential Equations and Linear Algebra, Final Exam I). The average in the class for this problem was 34.3%.