Eigenvalues and Diagonalization

EXAMPLE: Let

$$A = \begin{bmatrix} 7 & 4 \\ -3 & -1 \end{bmatrix}$$

and

$$\mathbf{x}_1 = \begin{bmatrix} 2 \\ -3 \end{bmatrix}, \qquad \mathbf{x}_2 = \begin{bmatrix} -2 \\ 1 \end{bmatrix}, \qquad \mathbf{x}_3 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Find $A\mathbf{x}_1$, $A\mathbf{x}_2$, and $A\mathbf{x}_3$.

Solution: We have

$$A\mathbf{x}_1 = \begin{bmatrix} 7 & 4 \\ -3 & -1 \end{bmatrix} \begin{bmatrix} 2 \\ -3 \end{bmatrix} = \begin{bmatrix} 2 \\ -3 \end{bmatrix}$$

$$A\mathbf{x}_2 = \begin{bmatrix} 7 & 4 \\ -3 & -1 \end{bmatrix} \begin{bmatrix} -2 \\ 1 \end{bmatrix} = \begin{bmatrix} -10 \\ 5 \end{bmatrix}$$

$$A\mathbf{x}_3 = \begin{bmatrix} 7 & 4 \\ -3 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 11 \\ -4 \end{bmatrix}$$

EXAMPLE: Let

$$A = \begin{bmatrix} 3 & -2 \\ 1 & 0 \end{bmatrix}$$

Find all nonzero vectors $\mathbf{x} \in \mathbb{R}^2$ and all scalars λ such that

$$A\mathbf{x} = \lambda\mathbf{x}$$

Solution: Suppose there is a vector $\mathbf{x} \in \mathbb{R}^2$ and a scalar λ such that

$$A\mathbf{x} = \lambda\mathbf{x}$$

hence

$$A\mathbf{x} - \lambda\mathbf{x} = \mathbf{0}$$

$$A\mathbf{x} - \lambda I\mathbf{x} = \mathbf{0}$$

$$(A - \lambda I)\mathbf{x} = \mathbf{0} \tag{1}$$

So, we should find such λ that (1) has a nontrivial solution.

THEOREM: Let A be a square $n \times n$ matrix. Then the equation

$$A\mathbf{x} = \mathbf{0}$$

has a nontrivial solution if and only if

$$\det A = 0$$

Proof: The result immediately follows from Theorem 2.5 (Section 2.3) that says that the system

$$A\mathbf{x} = \mathbf{0} \text{ has a nontrivial solution if } \operatorname{rank}(A) < n$$

and Corollary 3.6 (Section 3.2) that says that

$$\operatorname{rank}\,(A) = n \text{ if and only if } \det \neq 0 \quad \blacksquare$$

By the Theorem above, (1) has a nontrivial solution if and only if

$$\det(A - \lambda I) = 0 \tag{2}$$

Note that

$$A - \lambda I = \begin{bmatrix} 3 & -2 \\ 1 & 0 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 3-\lambda & -2 \\ 1 & -\lambda \end{bmatrix}$$

therefore we can rewrite (2) as

$$\begin{vmatrix} 3-\lambda & -2 \\ 1 & -\lambda \end{vmatrix} = 0$$

Expanding this determinant, we obtain

$$-(3-\lambda)\lambda + 2 = 0 \quad \Longrightarrow \quad \lambda^2 - 3\lambda + 2 = 0$$

Solving this quadratic equation, we get

$$\lambda_1 = 1, \quad \lambda_2 = 2$$

<u>Conclusion</u>: The equation

$$A\mathbf{x} = \lambda \mathbf{x}$$

has a nonzero solution $\mathbf{x} \in \mathbb{R}^2$ if and only if

$$\lambda = 1 \quad \text{or} \quad 2$$

(a) Let $\lambda = 1$. To solve the homogeneous system

$$(A - \lambda I)\mathbf{x} = \mathbf{0}$$

we use row operations

$$\begin{bmatrix} 3-\lambda & -2 & 0 \\ 1 & -\lambda & 0 \end{bmatrix} = \begin{bmatrix} 2 & -2 & 0 \\ 1 & -1 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

hence

$$x_1 - x_2 = 0 \quad \Longrightarrow \quad x_1 = x_2$$

We get

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_2 \\ x_2 \end{bmatrix} = x_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

(b) Similarly, if $\lambda = 2$, then

$$\begin{bmatrix} 3-\lambda & -2 & 0 \\ 1 & -\lambda & 0 \end{bmatrix} = \begin{bmatrix} 1 & -2 & 0 \\ 1 & -2 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

hence

$$x_1 - 2x_2 = 0 \quad \Longrightarrow \quad x_1 = 2x_2$$

We get

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2x_2 \\ x_2 \end{bmatrix} = x_2 \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

DEFINITION: An **eigenvector** of an $n \times n$ matrix A is a nonzero vector $\mathbf{x}$ such that

$$A\mathbf{x} = \lambda\mathbf{x}$$

for some scalar λ. A scalar λ is called an **eigenvalue** of A.

REMARK: To find eigenvalues of A we should solve the following **characteristic equation**

$$\det(A - \lambda I) = 0$$

where I is the identity matrix.

EXAMPLE: Let A be the same as above. Then $\lambda = 1$ and $\lambda = 2$ are the eigenvalues of A and

$$t\begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad t\begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

are the eigenvectors of A, where t is any nonzero scalar.

DEFINITION: Let A be an $n \times n$ matrix and let λ be an eigenvalue. The set of all solutions of the equation

$$(A - \lambda I)\mathbf{x} = \mathbf{0}$$

is called the **eigenspace** of A corresponding to λ.

EXAMPLE: Let A be the same as above. Then

$$\left\{ t\begin{bmatrix} 1 \\ 1 \end{bmatrix} : t \text{ is any scalar} \right\}$$

is the eigenspace of A corresponding to $\lambda = 1$;

$$\left\{ t\begin{bmatrix} 2 \\ 1 \end{bmatrix} : t \text{ is any scalar} \right\}$$

is the eigenspace of A corresponding to $\lambda = 2$.

EXAMPLE: Let

$$A = \begin{bmatrix} 4 & -1 & 6 \\ 2 & 1 & 6 \\ 2 & -1 & 8 \end{bmatrix}$$

The eigenvalues are 2 and 9. Find the corresponding eigenspaces.

Solution:

(a) Let $\lambda = 2$. We use row operations: We use row operations:

$$\begin{bmatrix} 4-\lambda & -1 & 6 & 0 \\ 2 & 1-\lambda & 6 & 0 \\ 2 & -1 & 8-\lambda & 0 \end{bmatrix} = \begin{bmatrix} 2 & -1 & 6 & 0 \\ 2 & -1 & 6 & 0 \\ 2 & -1 & 6 & 0 \end{bmatrix} \sim \begin{bmatrix} 2 & -1 & 6 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & -1/2 & 3 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

hence

$$x_1 - \frac{1}{2}x_2 + 3x_3 = 0 \quad \Longrightarrow \quad x_1 = \frac{1}{2}x_2 - 3x_3$$

therefore

$$\mathbf{x}=\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}=\begin{bmatrix} \frac{1}{2}x_2-3x_3 \\ x_2 \\ x_3 \end{bmatrix}=\begin{bmatrix} \frac{1}{2}\cdot x_2 & + & (-3)\cdot x_3 \\ 1\cdot x_2 & + & 0\cdot x_3 \\ 0\cdot x_2 & + & 1\cdot x_3 \end{bmatrix}=x_2\begin{bmatrix} 1/2 \\ 1 \\ 0 \end{bmatrix}+x_3\begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix}$$

is the eigenvector of A (x_2 and x_3 are not both zero) and

$$H=\left\{t_1\begin{bmatrix} 1/2 \\ 1 \\ 0 \end{bmatrix}+t_2\begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix} \ : \ t_1,t_2\in\mathbb{R}\right\}$$

is the eigenspace of A corresponding to $\lambda=2$.

(b) Let $\lambda=9$. We use row operations:

$$\begin{aligned}\begin{bmatrix} 4-\lambda & -1 & 6 & 0 \\ 2 & 1-\lambda & 6 & 0 \\ 2 & -1 & 8-\lambda & 0 \end{bmatrix}=\begin{bmatrix} -5 & -1 & 6 & 0 \\ 2 & -8 & 6 & 0 \\ 2 & -1 & -1 & 0 \end{bmatrix}&\sim\begin{bmatrix} 2 & -8 & 6 & 0 \\ -5 & -1 & 6 & 0 \\ 2 & -1 & -1 & 0 \end{bmatrix}\\ &\sim\begin{bmatrix} 1 & -4 & 3 & 0 \\ -5 & -1 & 6 & 0 \\ 2 & -1 & -1 & 0 \end{bmatrix}\\ &\sim\begin{bmatrix} 1 & -4 & 3 & 0 \\ 0 & -21 & 21 & 0 \\ 0 & 7 & -7 & 0 \end{bmatrix}\\ &\sim\begin{bmatrix} 1 & -4 & 3 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 1 & -1 & 0 \end{bmatrix}\\ &\sim\begin{bmatrix} 1 & -4 & 3 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}\\ &\sim\begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}\end{aligned}$$

hence

$$\begin{cases} x_1-x_3=0 \\ x_2-x_3=0 \end{cases} \implies \begin{cases} x_1=x_3 \\ x_2=x_3 \end{cases}$$

therefore

$$\mathbf{x}=\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}=\begin{bmatrix} x_3 \\ x_3 \\ x_3 \end{bmatrix}=x_3\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

is the eigenvector of A (x_3 is nonzero) and

$$H=\left\{t\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} : t\in\mathbb{R}\right\}$$

is the eigenspace of A corresponding to $\lambda=9$.

Diagonalization

EXAMPLE: Let

$$A = \begin{bmatrix} 7 & 4 \\ -3 & -1 \end{bmatrix}, \quad D = \begin{bmatrix} 1 & 0 \\ 0 & 5 \end{bmatrix}$$

Find a formula for A^k and D^k.

Solution:

(a) We have

$$D^2 = \begin{bmatrix} 1 & 0 \\ 0 & 5 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 5 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 5^2 \end{bmatrix}$$

$$D^3 = D^2 D = \begin{bmatrix} 1 & 0 \\ 0 & 5^2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 5 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 5^3 \end{bmatrix}$$

In general,

$$D^k = \begin{bmatrix} 1^k & 0 \\ 0 & 5^k \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 5^k \end{bmatrix}$$

(b) We have

$$A^2 = \begin{bmatrix} 7 & 4 \\ -3 & -1 \end{bmatrix} \begin{bmatrix} 7 & 4 \\ -3 & -1 \end{bmatrix} = \begin{bmatrix} 37 & 24 \\ -18 & -11 \end{bmatrix}$$

$$A^3 = \begin{bmatrix} 37 & 24 \\ -18 & -11 \end{bmatrix} \begin{bmatrix} 7 & 4 \\ -3 & -1 \end{bmatrix} = \begin{bmatrix} 187 & 124 \\ -93 & -61 \end{bmatrix}$$

$$A^4 = \begin{bmatrix} 187 & 124 \\ -93 & -61 \end{bmatrix} \begin{bmatrix} 7 & 4 \\ -3 & -1 \end{bmatrix} = \begin{bmatrix} 937 & 624 \\ -468 & -311 \end{bmatrix}$$

$$A^5 = \begin{bmatrix} 937 & 624 \\ -468 & -311 \end{bmatrix} \begin{bmatrix} 7 & 4 \\ -3 & -1 \end{bmatrix} = \begin{bmatrix} 4687 & 3124 \\ -2343 & -1561 \end{bmatrix}$$

DEFINITION: If A and B are $n \times n$ matrices, then A is **similar** to B if there is an invertible matrix P such that

$$P^{-1}AP = B$$

or, equivalently,

$$A = PBP^{-1}$$

EXAMPLE: Matrices

$$A = \begin{bmatrix} 7 & 4 \\ -3 & -1 \end{bmatrix} \quad \text{and} \quad D = \begin{bmatrix} 1 & 0 \\ 0 & 5 \end{bmatrix}$$

are similar, since

$$A = PDP^{-1}$$

where

$$P = \begin{bmatrix} -2 & -2 \\ 3 & 1 \end{bmatrix}$$

THEOREM: If $n \times n$ matrices A and B are similar, then they have the same characteristic polynomial and hence the same eigenvalues (with the same multiplicities).

Proof: If $B = P^{-1}AP$, then

$$\begin{aligned} B - \lambda I &= P^{-1}AP - \lambda P^{-1}P \\ &= P^{-1}(AP - \lambda P) \\ &= P^{-1}(AP - \lambda IP) = P^{-1}(A - \lambda I)P \end{aligned}$$

Using basic properties of determinants, we compute

$$\begin{aligned} \det(B - \lambda I) &= \det[P^{-1}(A - \lambda I)P] \\ &= \det(P^{-1}) \cdot \det(A - \lambda I) \cdot \det(P) \\ &= (\det P)^{-1} \cdot \det(A - \lambda I) \cdot \det(P) = \det(A - \lambda I) \end{aligned}$$

so

$$\det(B - \lambda I) = \det(A - \lambda I) \quad \blacksquare$$

EXAMPLE: Let A, D, and P be the same as above. Since $A = PDP^{-1}$, we have

$$\begin{aligned} A^2 &= PDP^{-1}PDP^{-1} \\ &= PD\underbrace{(P^{-1}P)}_{I}DP^{-1} \\ &= PDDP^{-1} = PD^2P^{-1} \end{aligned}$$

Similarly,

$$\begin{aligned} A^3 &= PD^3P^{-1} \\ A^4 &= PD^4P^{-1} \end{aligned}$$

In general,

$$A^k = PD^kP^{-1} = \begin{bmatrix} -2 & -2 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 5^k \end{bmatrix} \begin{bmatrix} 1/4 & 1/2 \\ -3/4 & -1/2 \end{bmatrix}$$

DEFINITION: A square matrix A is said to be **diagonalizable** if A is similar to a diagonal matrix, that is

$$A = PDP^{-1}$$

for some invertible matrix P and some diagonal matrix D.

EXAMPLE: Since

$$\begin{bmatrix} 7 & 4 \\ -3 & -1 \end{bmatrix} = \begin{bmatrix} -2 & -2 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 5 \end{bmatrix} \begin{bmatrix} -2 & -2 \\ 3 & 1 \end{bmatrix}^{-1}$$

it follows that $A = \begin{bmatrix} 7 & 4 \\ -3 & -1 \end{bmatrix}$ is diagonalizable.

THEOREM (The Diagonalization Theorem): An $n \times n$ matrix A is diagonalizable if and only if A has n eigenvectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ such that the matrix

$$P = [\mathbf{v}_1 \ \mathbf{v}_2 \ \ldots \ \mathbf{v}_n]$$

is invertible. In this case the diagonal entries of D are eigenvalues of A that correspond, respectively, to the eigenvectors in P.

EXAMPLE: Let A be the same as above. One can check that $\lambda = 1, 5$ are the eigenvalues of A and

$$\begin{bmatrix} -2 \\ 3 \end{bmatrix}, \quad \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

are the corresponding eigenvectors. Therefore

$$D = \begin{bmatrix} 1 & 0 \\ 0 & 5 \end{bmatrix} \quad \text{and} \quad P = \begin{bmatrix} -2 & -2 \\ 3 & 1 \end{bmatrix}$$

EXAMPLE: Determine if the following matrix is diagonalizable

$$A = \begin{bmatrix} 1 & 3 & 3 \\ -3 & -5 & -3 \\ 3 & 3 & 1 \end{bmatrix}$$

Solution: We first solve the following equation

$$\det(A - \lambda I) = \begin{vmatrix} 1-\lambda & 3 & 3 \\ -3 & -5-\lambda & -3 \\ 3 & 3 & 1-\lambda \end{vmatrix} = 0$$

Expanding this determinant, we obtain

$$-\lambda^3 - 3\lambda^2 + 4 = (1-\lambda)(\lambda+2)^2 = 0$$

hence

$$\lambda_1 = 1, \quad \lambda_2 = -2$$

are the eigenvalues of A, so

$$D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{bmatrix}$$

(a) Let $\lambda = 1$. We have

$$\begin{bmatrix} 1-\lambda & 3 & 3 & 0 \\ -3 & -5-\lambda & -3 & 0 \\ 3 & 3 & 1-\lambda & 0 \end{bmatrix} = \begin{bmatrix} 0 & 3 & 3 & 0 \\ -3 & -6 & -3 & 0 \\ 3 & 3 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 2 & 1 & 0 \\ 1 & 1 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 2 & 1 & 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

hence

$$\begin{cases} x_1 - x_3 = 0 \\ x_2 + x_3 = 0 \end{cases} \implies \begin{cases} x_1 = x_3 \\ x_2 = -x_3 \end{cases}$$

therefore

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_3 \\ -x_3 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

is the eigenvector of A, where x_3 is any nonzero scalar. In particular, $\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$ is an eigenvector of A.

(b) Let $\lambda = -2$. We have

$$\begin{bmatrix} 1-\lambda & 3 & 3 & 0 \\ -3 & -5-\lambda & -3 & 0 \\ 3 & 3 & 1-\lambda & 0 \end{bmatrix} = \begin{bmatrix} 3 & 3 & 3 & 0 \\ -3 & -3 & -3 & 0 \\ 3 & 3 & 3 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

hence

$$x_1 + x_2 + x_3 = 0 \implies x_1 = -x_2 - x_3$$

therefore

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -x_2 - x_3 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} (-1)\cdot x_2 & + & (-1)\cdot x_3 \\ 1\cdot x_2 & + & 0\cdot x_3 \\ 0\cdot x_2 & + & 1\cdot x_3 \end{bmatrix} = x_2 \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

is the eigenvector of A, where x_2, x_3 are any scalars (not both zero). In particular, $\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$ are eigenvectors of A.

So,

$$\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}, \quad \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

are eigenvectors of A. Moreover,

$$\begin{vmatrix} 1 & -1 & -1 \\ -1 & 1 & 0 \\ 1 & 0 & 1 \end{vmatrix} \neq 0$$

Therefore

$$P = \begin{bmatrix} 1 & -1 & -1 \\ -1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

and

$$\begin{bmatrix} 1 & 3 & 3 \\ -3 & -5 & -3 \\ 3 & 3 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -1 & -1 \\ -1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{bmatrix} \begin{bmatrix} 1 & -1 & -1 \\ -1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}^{-1}$$

EXAMPLE: Determine if the following matrix is diagonalizable

$$A = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Solution 1: Note that A is diagonal and therefore it is diagonalizable. Indeed, putting

$$P = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \quad \text{with} \quad ad - bc \neq 0 \qquad \text{and} \qquad D = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

we get $A = PDP^{-1}$, since

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1}$$

Solution 2: We have

$$\det(A - \lambda I) = \begin{vmatrix} 0 - \lambda & 0 \\ 0 & 0 - \lambda \end{vmatrix} = \begin{vmatrix} -\lambda & 0 \\ 0 & -\lambda \end{vmatrix} = 0 \quad \Longrightarrow \quad (-\lambda)^2 - 0^2 = 0 \quad \Longrightarrow \quad \lambda^2 = 0$$

hence

$$\lambda = 0$$

is the eigenvalue of A. So

$$D = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

We now find the eigenvectors. We have

$$\begin{bmatrix} 0 - \lambda & 0 & 0 \\ 0 & 0 - \lambda & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

therefore the corresponding equation is

$$0 = 0$$

In other words, *any* nonzero vector $\mathbf{x}$ is an eigenvector of A. Putting

$$P = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \quad \text{with} \quad ad - bc \neq 0$$

we come up with the same result.

EXAMPLE: Determine if the following matrix is diagonalizable

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

Solution: We have

$$\det(A - \lambda I) = \begin{vmatrix} 0 - \lambda & 1 \\ 0 & 0 - \lambda \end{vmatrix} = \begin{vmatrix} -\lambda & 1 \\ 0 & -\lambda \end{vmatrix} = 0 \quad \Longrightarrow \quad (-\lambda)^2 - 0 = 0 \quad \Longrightarrow \quad \lambda^2 = 0$$

hence

$$\lambda = 0$$

is the eigenvalue of A. So

$$D = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

We now find the eigenvectors. We have

$$\begin{bmatrix} 0-\lambda & 1 & 0 \\ 0 & 0-\lambda & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

therefore the corresponding equation is

$$x_2 = 0$$

Hence

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ 0 \end{bmatrix}$$

is the eigenvector of A, where x_1 is any nonzero scalar. Since the determinant of

$$\begin{bmatrix} c_1 & c_2 \\ 0 & 0 \end{bmatrix}$$

is equal to zero for any c_1, c_2, it follows that A is not diagonalizable.