Further Properties of the Determinant

THEOREM 3.7: If A and B are both $n \times n$ matrices, then

$$\det(AB) = \det A \det B$$

Proof: We distinguish two cases:

Case A: First, suppose A is singular. Then $\det A = 0$ by Theorem 3.5 from Section 3.2. If $\det(AB) = 0$, then $\det(AB) = \det A \det B$ and we will be done. We assume $\det(AB) \neq 0$ and get a contradiction. If $\det(AB) \neq 0$, $(AB)^{-1}$ exists, and

$$I_n = AB(AB)^{-1}$$

Hence, $B(AB)^{-1}$ is a right inverse for A. But then by Theorem 2.9 from Section 2.4, A^{-1} exists, contradicting the fact that A is singular.

Case B: Now suppose A is nonsingular. In the special case where $A = I_n$, we have $\det A = 1$, and so

$$\det(AB) = \det(I_nB) = \det B = 1 \cdot \det B = \det A \det B$$

Finally, if A is any other nonsingular matrix, then A is row equivalent to I_n, so there is a sequence $R_1, R_2, \ldots, R_k$ of row operations such that

$$R_k(\ldots(R_2(R_1(I_n)))\ldots) = A$$

(These are the inverses of the row operations that row reduce A to I_n.) Now, each row operation R_i has an associated real number r_i, so that applying R_i to a matrix multiplies its determinant by r_i (as in Theorem 3.3 from Section 3.2). Hence,

$$\begin{aligned}
\det(AB) &= \det(R_k(\ldots(R_2(R_1(I_n)))\ldots)B) \\
&= \det(R_k(\ldots(R_2(R_1(I_nB)))\ldots)) && \text{by Theorem 2.1 (Section 2.1), part (2)} \\
&= r_k \ldots r_2 r_1 \det(I_nB) && \text{by Theorem 3.3} \\
&= r_k \ldots r_2 r_1 \det(I_n)\det(B) && \text{by the } I_n \text{ special case} \\
&= \det(R_k(\ldots(R_2(R_1(I_n)))\ldots))\det(B) && \text{by Theorem 3.3} \\
&= \det A \det B
\end{aligned}$$

COROLLARY 3.8: If A is nonsingular, then

$$\det(A^{-1}) = (\det A)^{-1}$$

Proof: If A is nonsingular, then $AA^{-1} = I_n$. By Theorem 3.7,

$$\det A \det(A^{-1}) = \det(AA^{-1}) = \det I_n = 1$$

so

$$\det(A^{-1}) = (\det A)^{-1} \quad \blacksquare$$

THEOREM 3.9: If A is an $n \times n$ matrix, then

$$\det A = \det A^T$$

Cramer's Rule

DEFINITION: For any $n \times n$ matrix A and any $\mathbf{b}$ in R^n, let $A_i(\mathbf{b})$ be the matrix obtained from A by replacing column i by the vector $\mathbf{b}$.

EXAMPLE: Let

$$A = \begin{bmatrix} 2 & 1 & 3 \\ 1 & 0 & 4 \\ 0 & 0 & 5 \end{bmatrix} \quad \text{and} \quad \mathbf{b} = \begin{bmatrix} 3 \\ 8 \\ 9 \end{bmatrix}$$

Then

$$A_1(\mathbf{b}) = \begin{bmatrix} 3 & 1 & 3 \\ 8 & 0 & 4 \\ 9 & 0 & 5 \end{bmatrix} \qquad A_2(\mathbf{b}) = \begin{bmatrix} 2 & 3 & 3 \\ 1 & 8 & 4 \\ 0 & 9 & 5 \end{bmatrix} \qquad A_3(\mathbf{b}) = \begin{bmatrix} 2 & 1 & 3 \\ 1 & 0 & 8 \\ 0 & 0 & 9 \end{bmatrix}$$

THEOREM (CRAMER'S RULE): Let A be an invertible $n \times n$ matrix. For any $\mathbf{b}$ in $\mathbb{R}^n$, the unique solution $\mathbf{x}$ of $A\mathbf{x} = \mathbf{b}$ has entries given by

$$x_i = \frac{\det A_i(\mathbf{b})}{\det A}, \quad i = 1, 2, \ldots, n.$$

EXAMPLE: Solve using Cramer's rule

$$\begin{cases} x_1 - 2x_2 = 1 \\ 3x_1 + 4x_2 = -7 \end{cases}$$

Solution: We have

$$x_1 = \frac{\begin{vmatrix} 1 & -2 \\ -7 & 4 \end{vmatrix}}{\begin{vmatrix} 1 & -2 \\ 3 & 4 \end{vmatrix}} = \frac{4 - 14}{4 - (-6)} = \frac{-10}{10} = -1 \qquad \text{and} \qquad x_2 = \frac{\begin{vmatrix} 1 & 1 \\ 3 & -7 \end{vmatrix}}{\begin{vmatrix} 1 & -2 \\ 3 & 4 \end{vmatrix}} = \frac{-7 - 3}{10} = \frac{-10}{10} = -1$$

Formula for A^{-1}

DEFINITION: For any $n \times n$ matrix A, let A_{ij} be the submatrix of A, formed by deleting row i and column j.

EXAMPLE: Let $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 0 \end{bmatrix}$. Then

$$A_{11} = \begin{bmatrix} 5 & 6 \\ 8 & 0 \end{bmatrix} \quad A_{12} = \begin{bmatrix} 4 & 6 \\ 7 & 0 \end{bmatrix} \quad A_{13} = \begin{bmatrix} 4 & 5 \\ 7 & 8 \end{bmatrix}$$

$$A_{21} = \begin{bmatrix} 2 & 3 \\ 8 & 0 \end{bmatrix} \quad A_{22} = \begin{bmatrix} 1 & 3 \\ 7 & 0 \end{bmatrix} \quad A_{23} = \begin{bmatrix} 1 & 2 \\ 7 & 8 \end{bmatrix}$$

$$A_{31} = \begin{bmatrix} 2 & 3 \\ 5 & 6 \end{bmatrix} \quad A_{32} = \begin{bmatrix} 1 & 3 \\ 4 & 6 \end{bmatrix} \quad A_{33} = \begin{bmatrix} 1 & 2 \\ 4 & 5 \end{bmatrix}$$

THEOREM (AN INVERSE FORMULA): Let A be an invertible $n \times n$ matrix. Then

$$A^{-1} = \frac{1}{\det A} \begin{bmatrix} C_{11} & C_{12} & \dots & C_{1n} \\ C_{21} & C_{22} & \dots & C_{2n} \\ \dots & \dots & \dots & \dots \\ C_{n1} & C_{n2} & \dots & C_{nn} \end{bmatrix}^T$$

where

$$C_{ij} = (-1)^{i+j} \det A_{ij}$$

EXAMPLE: Let $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 0 \end{bmatrix}$. Find A^{-1}.

Solution 1:

Step 1: One can verify that $\det A = 27$.

Step 2: We have

$$A_{11} = \begin{bmatrix} 5 & 6 \\ 8 & 0 \end{bmatrix} \quad A_{12} = \begin{bmatrix} 4 & 6 \\ 7 & 0 \end{bmatrix} \quad A_{13} = \begin{bmatrix} 4 & 5 \\ 7 & 8 \end{bmatrix}$$

$$\det A_{11} = -48 \quad \det A_{12} = -42 \quad \det A_{13} = -3$$

$$C_{11} = -48 \qquad C_{12} = 42 \qquad C_{13} = -3$$

$$A_{21} = \begin{bmatrix} 2 & 3 \\ 8 & 0 \end{bmatrix} \quad A_{22} = \begin{bmatrix} 1 & 3 \\ 7 & 0 \end{bmatrix} \quad A_{23} = \begin{bmatrix} 1 & 2 \\ 7 & 8 \end{bmatrix}$$

$$\det A_{21} = -24 \quad \det A_{22} = -21 \quad \det A_{23} = -6$$

$$C_{21} = 24 \qquad C_{22} = -21 \qquad C_{23} = 6$$

$$A_{31} = \begin{bmatrix} 2 & 3 \\ 5 & 6 \end{bmatrix} \quad A_{32} = \begin{bmatrix} 1 & 3 \\ 4 & 6 \end{bmatrix} \quad A_{33} = \begin{bmatrix} 1 & 2 \\ 4 & 5 \end{bmatrix}$$

$$\det A_{31} = -3 \quad \det A_{32} = -6 \quad \det A_{33} = -3$$

$$C_{31} = -3 \qquad C_{32} = 6 \qquad C_{33} = -3$$

Step 3: We have

$$A^{-1} = \frac{1}{27} \begin{bmatrix} -48 & 42 & -3 \\ 24 & -21 & 6 \\ -3 & 6 & -3 \end{bmatrix}^T = \frac{1}{27} \begin{bmatrix} -48 & 24 & -3 \\ 42 & -21 & 6 \\ -3 & 6 & -3 \end{bmatrix}$$

$$= \begin{bmatrix} -16/9 & 8/9 & -1/9 \\ 14/9 & -7/9 & 2/9 \\ -1/9 & 2/9 & -1/9 \end{bmatrix}$$

Solution 2: We have

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 4 & 5 & 6 & 0 & 1 & 0 \\ 7 & 8 & 0 & 0 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & -3 & -6 & -4 & 1 & 0 \\ 0 & -6 & -21 & -7 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & -3 & -6 & -4 & 1 & 0 \\ 0 & 0 & -9 & 1 & -2 & 1 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & 2 & 4/3 & -1/3 & 0 \\ 0 & 0 & 1 & -1/9 & 2/9 & -1/9 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 2 & 0 & 4/3 & -2/3 & 1/3 \\ 0 & 1 & 0 & 14/9 & -7/9 & 2/9 \\ 0 & 0 & 1 & -1/9 & 2/9 & -1/9 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -16/9 & 8/9 & -1/9 \\ 0 & 1 & 0 & 14/9 & -7/9 & 2/9 \\ 0 & 0 & 1 & -1/9 & 2/9 & -1/9 \end{array}\right]$$

and the same result follows.