Determinants and Row Reduction

THEOREM 3.2: If A is a triangular matrix, then $\det A$ is the product of the entries on the main diagonal of A.

EXAMPLE:

$$\begin{vmatrix} 3 & -7 & 8 & 9 & -6 \\ 0 & 2 & -5 & 7 & 3 \\ 0 & 0 & 1 & 5 & 0 \\ 0 & 0 & 0 & 4 & -1 \\ 0 & 0 & 0 & 0 & 1 \end{vmatrix} = 3 \cdot 2 \cdot 1 \cdot 4 \cdot 1 = 24$$

THEOREM: We have $\det A = 0$

(a) if A contains a zero-row or zero-column.

EXAMPLE: $\begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 0 & 0 & 0 \end{vmatrix} = 0$

(b) if A contains two equal rows or columns.

EXAMPLE: $\begin{vmatrix} 1 & 2 & 1 \\ 2 & 5 & 2 \\ 2 & 0 & 2 \end{vmatrix} = 0$

(c) if some row (column) of A is a multiple of some other row (column) of A.

EXAMPLE: $\begin{vmatrix} 1 & 2 & 1 \\ -2 & -4 & -2 \\ 3 & 5 & 7 \end{vmatrix} = 0$

THEOREM 3.3: Let A be a square matrix.

(I) If one row (column) of A is multiplied by k to produce B, then

$$\det B = k \det A$$

EXAMPLE: $100 \begin{vmatrix} 1 & 3 \\ 1 & 2 \end{vmatrix} = \begin{vmatrix} 100 & 300 \\ 1 & 2 \end{vmatrix} = \begin{vmatrix} 1 & 3 \\ 100 & 200 \end{vmatrix} = \begin{vmatrix} 100 & 3 \\ 100 & 2 \end{vmatrix} = \begin{vmatrix} 1 & 300 \\ 1 & 200 \end{vmatrix} = \begin{vmatrix} 10 & 30 \\ 10 & 20 \end{vmatrix}$

REMARK: Note that $\begin{vmatrix} 10 & 30 \\ 10 & 20 \end{vmatrix} = 100 \begin{vmatrix} 1 & 3 \\ 1 & 2 \end{vmatrix}$, but $\begin{bmatrix} 10 & 30 \\ 10 & 20 \end{bmatrix} = 10 \begin{bmatrix} 1 & 3 \\ 1 & 2 \end{bmatrix}$.

(II) If a multiple of one row (column) of A is added to another row (column) to produce a matrix B, then

$$\det A = \det B$$

EXAMPLE: $\begin{vmatrix} 1 & 1 & 2 \\ 1 & 2 & 3 \\ 3 & 3 & 8 \end{vmatrix} = \begin{vmatrix} 1 & 1 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 2 \end{vmatrix} = 2$

(III) If two rows (columns) of A are interchanged to produce B, then

$$\det A = -\det B$$

EXAMPLE: $\begin{vmatrix} 1 & 1 & 2 \\ 1 & 2 & 3 \\ 3 & 3 & 8 \end{vmatrix} = - \begin{vmatrix} 3 & 3 & 8 \\ 1 & 2 & 3 \\ 1 & 1 & 2 \end{vmatrix} = \begin{vmatrix} 3 & 8 & 3 \\ 1 & 3 & 2 \\ 1 & 2 & 1 \end{vmatrix}$

EXAMPLE: Find

$$\begin{vmatrix} 1 & 3 & 5 & 4 \\ 2 & -3 & 1 & -1 \\ -1 & 2 & -1 & 0 \\ 2 & 2 & 5 & 3 \end{vmatrix}$$

Solution: We have

$$\begin{vmatrix} 1 & 3 & 5 & 4 \\ 2 & -3 & 1 & -1 \\ -1 & 2 & -1 & 0 \\ 2 & 2 & 5 & 3 \end{vmatrix} = \begin{vmatrix} 1 & 3 & 5 & 4 \\ 0 & -9 & -9 & -9 \\ 0 & 5 & 4 & 4 \\ 0 & -4 & -5 & -5 \end{vmatrix} = (-9) \begin{vmatrix} 1 & 3 & 5 & 4 \\ 0 & 1 & 1 & 1 \\ 0 & 5 & 4 & 4 \\ 0 & -4 & -5 & -5 \end{vmatrix}$$

$$= (-9) \begin{vmatrix} 1 & 3 & 5 & 4 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & -1 & -1 \\ 0 & 0 & -1 & -1 \end{vmatrix}$$

Since the last two rows are equal, the determinant is equal to 0.

THEOREM 3.5: An $n \times n$ matrix A is nonsingular if and only if $\det A \neq 0$.

Proof: Let D be the unique (Theorem 2.4 from Section 2.3) matrix in reduced row echelon form for A. Now, using Theorem 3.3, we see that a single row operation of type (I), (II), or (III) cannot convert a matrix having a nonzero determinant to a matrix having a zero determinant. Indeed,

(i) a single row operation of type (II) does not change the determinant;

(ii) a single row operation of type (III) switches the sign of the determinant, therefore it can't convert a matrix having a nonzero determinant to a matrix having a zero determinant;

(iii) a single row operation of type (I) can't convert a matrix having a nonzero determinant to a matrix having a zero determinant, since c is a *nonzero* constant.

So, because A is converted to D using a finite number of such row operations, Theorem 3.3 assures us that $\det A$ and $\det D$ are either both zero or both nonzero.

Now, if A is nonsingular (which implies $D = I_n$), we know that $\det D = 1 \neq 0$ and therefore $\det A \neq 0$, and we have completed half of the proof.

For the other half, assume that $\det A \neq 0$. Then $\det D \neq 0$. Because D is a square matrix with a staircase pattern of pivots, it is upper triangular. Because $\det D \neq 0$, Theorem 3.2 asserts that all main diagonal entries of D are nonzero. Hence, they are all pivots, and $D = I_n$. Therefore, row reduction transforms A to I_n, so A is nonsingular. ■

COROLLARY 3.6: Let A be an $n \times n$ matrix. Then rank $(A) = n$ if and only if $\det A \neq 0$.