

# Determinants and Row Reduction

THEOREM 3.2: If  $A$  is a triangular matrix, then  $\det A$  is the product of the entries on the main diagonal of  $A$ .

EXAMPLE:

$$\begin{vmatrix} 3 & -7 & 8 & 9 & -6 \\ 0 & 2 & -5 & 7 & 3 \\ 0 & 0 & 1 & 5 & 0 \\ 0 & 0 & 0 & 4 & -1 \\ 0 & 0 & 0 & 0 & 1 \end{vmatrix} = 3 \cdot 2 \cdot 1 \cdot 4 \cdot 1 = 24$$

THEOREM: We have  $\det A = 0$

(a) if  $A$  contains a zero-row or zero-column.

$$\text{EXAMPLE: } \begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 0 & 0 & 0 \end{vmatrix} = 0$$

(b) if  $A$  contains two equal rows or columns.

$$\text{EXAMPLE: } \begin{vmatrix} 1 & 2 & 1 \\ 2 & 5 & 2 \\ 2 & 0 & 2 \end{vmatrix} = 0$$

(c) if some row (column) of  $A$  is a multiple of some other row (column) of  $A$ .

$$\text{EXAMPLE: } \begin{vmatrix} 1 & 2 & 1 \\ -2 & -4 & -2 \\ 3 & 5 & 7 \end{vmatrix} = 0$$

THEOREM 3.3: Let  $A$  be a square matrix.

(I) If one row (column) of  $A$  is multiplied by  $k$  to produce  $B$ , then

$$\det B = k \det A$$

$$\text{EXAMPLE: } 100 \begin{vmatrix} 1 & 3 \\ 1 & 2 \end{vmatrix} = \begin{vmatrix} 100 & 300 \\ 1 & 2 \end{vmatrix} = \begin{vmatrix} 1 & 3 \\ 100 & 200 \end{vmatrix} = \begin{vmatrix} 100 & 3 \\ 100 & 2 \end{vmatrix} = \begin{vmatrix} 1 & 300 \\ 1 & 200 \end{vmatrix}$$

(II) If a multiple of one row (column) of  $A$  is added to another row (column) to produce a matrix  $B$ , then

$$\det A = \det B$$

$$\text{EXAMPLE: } \begin{vmatrix} 1 & 1 & 2 \\ 1 & 2 & 3 \\ 3 & 3 & 8 \end{vmatrix} = \begin{vmatrix} 1 & 1 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 2 \end{vmatrix} = 2$$

(III) If two rows (columns) of  $A$  are interchanged to produce  $B$ , then

$$\det A = -\det B$$

EXAMPLE: 
$$\begin{vmatrix} 1 & 1 & 2 \\ 1 & 2 & 3 \\ 3 & 3 & 8 \end{vmatrix} = -\begin{vmatrix} 3 & 3 & 8 \\ 1 & 2 & 3 \\ 1 & 1 & 2 \end{vmatrix} = \begin{vmatrix} 3 & 8 & 3 \\ 1 & 3 & 2 \\ 1 & 2 & 1 \end{vmatrix}$$

EXAMPLE: Find

$$\begin{vmatrix} 1 & 3 & 5 & 4 \\ 2 & -3 & 1 & -1 \\ -1 & 2 & -1 & 0 \\ 2 & 2 & 5 & 3 \end{vmatrix}$$

Solution: We have

$$\begin{aligned} \begin{vmatrix} 1 & 3 & 5 & 4 \\ 2 & -3 & 1 & -1 \\ -1 & 2 & -1 & 0 \\ 2 & 2 & 5 & 3 \end{vmatrix} &= \begin{vmatrix} 1 & 3 & 5 & 4 \\ 0 & -9 & -9 & -9 \\ 0 & 5 & 4 & 4 \\ 0 & -4 & -5 & -5 \end{vmatrix} = (-9) \begin{vmatrix} 1 & 3 & 5 & 4 \\ 0 & 1 & 1 & 1 \\ 0 & 5 & 4 & 4 \\ 0 & -4 & -5 & -5 \end{vmatrix} \\ &= (-9) \begin{vmatrix} 1 & 3 & 5 & 4 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & -1 & -1 \\ 0 & 0 & -1 & -1 \end{vmatrix} \end{aligned}$$

Since the last two rows are equal, the determinant is equal to 0.

**THEOREM 3.5:** An  $n \times n$  matrix  $A$  is nonsingular if and only if  $\det A \neq 0$ .

Proof: Let  $D$  be the unique matrix in reduced row echelon form for  $A$ . Now, using Theorem 3.3, we see that a single row operation of type (I), (II), or (III) cannot convert a matrix having a nonzero determinant to a matrix having a zero determinant. Indeed,

- (i) a single row operation of type (II) does not change the determinant;
- (ii) a single row operation of type (III) switches the sign of the determinant, therefore it can't convert a matrix having a nonzero determinant to a matrix having a zero determinant;
- (iii) a single row operation of type (I) can't convert a matrix having a nonzero determinant to a matrix having a zero determinant, since  $c$  is a *nonzero* constant.

So, because  $A$  is converted to  $D$  using a finite number of such row operations, Theorem 3.3 assures us that  $\det A$  and  $\det D$  are either both zero or both nonzero.

Now, if  $A$  is nonsingular (which implies  $D = I_n$ ), we know that  $\det D = 1 \neq 0$  and therefore  $\det A \neq 0$ , and we have completed half of the proof.

For the other half, assume that  $\det A \neq 0$ . Then  $\det D \neq 0$ . Because  $D$  is a square matrix with a staircase pattern of pivots, it is upper triangular. Because  $\det D \neq 0$ , Theorem 3.2 asserts that all main diagonal entries of  $D$  are nonzero. Hence, they are all pivots, and  $D = I_n$ . Therefore, row reduction transforms  $A$  to  $I_n$ , so  $A$  is nonsingular. ■

**COROLLARY 3.6:** Let  $A$  be an  $n \times n$  matrix. Then  $\text{rank}(A) = n$  if and only if  $\det A \neq 0$ .