Introduction to Determinants

Let

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{bmatrix}$$

DEFINITION: The **determinant** of an $n \times n$ matrix A is the following sum:

$$\det A = a_{11} \det A_{11} - a_{12} \det A_{12} + a_{13} \det A_{13} - \ldots + (-1)^{n+1} a_{1n} \det A_{1n}$$

where A_{1j} are submatrices formed by deleting from A the first row and jth column.

EXAMPLE: Find $\begin{vmatrix} 1 & 0 & 0 & 1 \\ 0 & 2 & 0 & -1 \\ -1 & 0 & 3 & 0 \\ 0 & 1 & 1 & 1 \end{vmatrix}$.

Solution: We have

$$\begin{vmatrix} 1 & 0 & 0 & 1 \\ 0 & 2 & 0 & -1 \\ -1 & 0 & 3 & 0 \\ 0 & 1 & 1 & 1 \end{vmatrix} = \begin{vmatrix} 2 & 0 & -1 \\ 0 & 3 & 0 \\ 1 & 1 & 1 \end{vmatrix} - \begin{vmatrix} 0 & 2 & 0 \\ -1 & 0 & 3 \\ 0 & 1 & 1 \end{vmatrix}$$

Since

$$\begin{vmatrix} 2 & 0 & -1 \\ 0 & 3 & 0 \\ 1 & 1 & 1 \end{vmatrix} = 2 \begin{vmatrix} 3 & 0 \\ 1 & 1 \end{vmatrix} + (-1) \begin{vmatrix} 0 & 3 \\ 1 & 1 \end{vmatrix} = 2(3-0) + (-1)(0-3) = 9$$

and

$$\begin{vmatrix} 0 & 2 & 0 \\ -1 & 0 & 3 \\ 0 & 1 & 1 \end{vmatrix} = -2 \begin{vmatrix} -1 & 3 \\ 0 & 1 \end{vmatrix} = -2(-1) = 2$$

it follows that the determinant is equal to $9 - 2 = 7$.

REMARK: One can find $\begin{vmatrix} 2 & 0 & -1 \\ 0 & 3 & 0 \\ 1 & 1 & 1 \end{vmatrix}$ in an easier way expanding by row two:

$$\begin{vmatrix} 2 & 0 & -1 \\ 0 & 3 & 0 \\ 1 & 1 & 1 \end{vmatrix} = 3 \begin{vmatrix} 2 & -1 \\ 1 & 1 \end{vmatrix} = 3(2 + (-1)) = 9$$

THEOREM: If A is a 2×2 matrix, the area of the parallelogram determined by the columns of A is $|\det A|$. If A is a 3×3 matrix, the volume of the parallelepiped determined by the columns of A is $|\det A|$.