Inverses of Matrices

DEFINITION: An $n \times n$ matrix A is said to be invertible if there is an $n \times n$ matrix C such that

$$CA = I \quad \text{and} \quad AC = I$$

In this case, C is an **inverse** of A and is denoted by A^{-1}. So,

$$A^{-1}A = I \quad \text{and} \quad AA^{-1} = I$$

DEFINITION: A square matrix is **singular** if and only if it does not have an inverse. A square matrix is **nonsingular** if and only if it has an inverse.

EXAMPLE: Let $A = \begin{bmatrix} 2 & 5 \\ -3 & -7 \end{bmatrix}$. Then $A^{-1} = \begin{bmatrix} -7 & -5 \\ 3 & 2 \end{bmatrix}$. In fact, we have

$$AA^{-1} = \begin{bmatrix} 2 & 5 \\ -3 & -7 \end{bmatrix} \begin{bmatrix} -7 & -5 \\ 3 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

and

$$A^{-1}A = \begin{bmatrix} -7 & -5 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} 2 & 5 \\ -3 & -7 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

THEOREM 2.9: Let A and B be $n \times n$ matrices. If either product AB or BA equals I_n, then the other product also equals I_n, and A and B are inverses of each other.

THEOREM 2.10 (Uniqueness of Inverse Matrix): If B and C are both inverses of an $n \times n$ matrix A, then $B = C$.

Proof: We have $B = BI = B(AC) = (BA)C = IC = C$ ■

THEOREM 2.13: Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. If $ad - bc \neq 0$, then A is invertible and

$$A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

If $ad - bc = 0$, then A is not invertible.

EXAMPLE: Solve the following system of equations $\begin{cases} x_1 - 2x_2 = 0 \\ x_1 + 4x_2 = 6 \end{cases}$.

Solution: We have

$$A\mathbf{x} = B$$

$$A^{-1}(A\mathbf{x}) = A^{-1}B$$

But $A^{-1}(A\mathbf{x}) = (A^{-1}A)\mathbf{x} = I\mathbf{x} = \mathbf{x}$, therefore

$$\mathbf{x} = A^{-1}B$$

hence

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 & -2 \\ 1 & 4 \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 6 \end{bmatrix} = \frac{1}{6} \begin{bmatrix} 4 & 2 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 6 \end{bmatrix} = \frac{1}{6} \begin{bmatrix} 12 \\ 6 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

so $x_1 = 2$ and $x_2 = 1$.

PROPERTIES: Let A and B be invertible $n \times n$ matrices. Then

(a) $(A^{-1})^{-1} = A$ (b) $(AB)^{-1} = B^{-1}A^{-1}$ (c) $(A^T)^{-1} = (A^{-1})^T$

Proof:

(a) We have

$$\begin{aligned} I &= (A^{-1})^{-1}A^{-1} \\ IA &= \left[(A^{-1})^{-1}A^{-1}\right] A \\ A &= (A^{-1})^{-1}\left[A^{-1}A\right] \\ &= (A^{-1})^{-1}I \\ &= (A^{-1})^{-1} \end{aligned}$$

Here is another way to prove it. Since $AA^{-1} = I$ and $A^{-1}A = I$, it follows that A is the inverse of A^{-1}. That is, $(A^{-1})^{-1} = A$.

(b) We have

$$\begin{aligned} I &= (AB)^{-1}(AB) \\ I(B^{-1}A^{-1}) &= \left[(AB)^{-1}(AB)\right](B^{-1}A^{-1}) \\ B^{-1}A^{-1} &= (AB)^{-1}\left[(AB)(B^{-1}A^{-1})\right] \\ &= (AB)^{-1}\left[A(B(B^{-1}A^{-1}))\right] \\ &= (AB)^{-1}\left[A((BB^{-1})A^{-1})\right] \\ &= (AB)^{-1}\left[A(IA^{-1})\right] \\ &= (AB)^{-1}\left[AA^{-1}\right] \\ &= (AB)^{-1}I \\ &= (AB)^{-1} \end{aligned}$$

Here is another way to prove it. We have

$$(AB)(B^{-1}A^{-1}) = A[B(B^{-1}A^{-1})] = A[(BB^{-1})A^{-1}] = A[IA^{-1}] = AA^{-1} = I$$

So, $B^{-1}A^{-1}$ is the inverse of AB by Theorem 2.9. That is, $B^{-1}A^{-1} = (AB)^{-1}$.

(c) We have

$$\begin{aligned} I &= (A^T)^{-1}A^T \\ I(A^{-1})^T &= \left[(A^T)^{-1}A^T\right](A^{-1})^T \\ (A^{-1})^T &= (A^T)^{-1}\left[A^T(A^{-1})^T\right] \\ &= (A^T)^{-1}(A^{-1}A)^T \\ &= (A^T)^{-1}I^T \\ &= (A^T)^{-1}I \\ &= (A^T)^{-1} \end{aligned}$$

Here is another way to prove it. We have

$$A^T(A^{-1})^T = (A^{-1}A)^T = I^T = I$$

So, $(A^{-1})^T$ is the inverse of A^T by Theorem 2.9. That is, $(A^{-1})^T = (A^T)^{-1}$.

ALGORITHM FOR FINDING A^{-1}:

1. Row reduce the augmented matrix $[A\ I]$.
2. If A is row equivalent to I, then $[A\ I]$ is row equivalent to $[I\ A^{-1}]$.
3. Otherwise, A does not have an inverse.

EXAMPLE: Let

$$A = \begin{bmatrix} 3 & -1 & 1 \\ -1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

Find A^{-1}.

Solution: We have

$$\left[\begin{array}{rrr|rrr} 3 & -1 & 1 & 1 & 0 & 0 \\ -1 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{rrr|rrr} 1 & 1 & 1 & 1 & 2 & 0 \\ -1 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{rrr|rrr} 1 & 1 & 1 & 1 & 2 & 0 \\ 0 & 2 & 1 & 1 & 3 & 0 \\ 0 & -1 & 0 & -1 & -2 & 1 \end{array}\right]$$

$$\sim \left[\begin{array}{rrr|rrr} 1 & 1 & 1 & 1 & 2 & 0 \\ 0 & 1 & 1 & 0 & 1 & 1 \\ 0 & -1 & 0 & -1 & -2 & 1 \end{array}\right]$$

$$\sim \left[\begin{array}{rrr|rrr} 1 & 1 & 1 & 1 & 2 & 0 \\ 0 & 1 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & -1 & -1 & 2 \end{array}\right]$$

$$\sim \left[\begin{array}{rrr|rrr} 1 & 0 & 0 & 1 & 1 & -1 \\ 0 & 1 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & -1 & -1 & 2 \end{array}\right]$$

$$\sim \left[\begin{array}{rrr|rrr} 1 & 0 & 0 & 1 & 1 & -1 \\ 0 & 1 & 0 & 1 & 2 & -1 \\ 0 & 0 & 1 & -1 & -1 & 2 \end{array}\right]$$

therefore

$$A^{-1} = \begin{bmatrix} 1 & 1 & -1 \\ 1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix}$$