Matrix Multiplication

DEFINITION: If A is an $m \times n$ matrix, and if B is an $n \times p$ matrix with columns $\mathbf{b}_1, \ldots, \mathbf{b}_p$, then the product AB is the $m \times p$ matrix whose columns are $A\mathbf{b}_1, \ldots, A\mathbf{b}_p$. That is,

$$AB = A[\mathbf{b}_1 \;\; \mathbf{b}_2 \;\; \ldots \;\; \mathbf{b}_p] = [A\mathbf{b}_1 \;\; A\mathbf{b}_2 \;\; \ldots \;\; A\mathbf{b}_p]$$

ROW-COLUMN RULE FOR COMPUTING AB: If the product AB is defined, then the entry in row i and column j of AB is the sum of the products of corresponding entries from row i of A and column j of B. If $(AB)_{ij}$ denotes the (i,j)-entry in AB, and if A is an $m \times n$ matrix, then

$$(AB)_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \ldots + a_{in}b_{nj}$$

EXAMPLE: Let $A = \begin{bmatrix} 2 & 3 \\ 1 & -5 \end{bmatrix}$ and $B = \begin{bmatrix} 4 & 3 & 6 \\ 1 & -2 & 3 \end{bmatrix}$, then

$$AB = \begin{bmatrix} 2\cdot 4 + 3\cdot 1 & 2\cdot 3 + 3\cdot(-2) & 2\cdot 6 + 3\cdot 3 \\ 1\cdot 4 + (-5)\cdot 1 & 1\cdot 3 + (-5)\cdot(-2) & 1\cdot 6 + (-5)\cdot 3 \end{bmatrix} = \begin{bmatrix} 11 & 0 & 21 \\ -1 & 13 & -9 \end{bmatrix}$$

Note that BA is undefined.

EXAMPLE:

Consider the following matrices:

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & 4 \end{bmatrix}, \qquad B = \begin{bmatrix} 1 & 0 \\ 2 & 1 \\ 3 & 2 \end{bmatrix}, \qquad C = \begin{bmatrix} 3 & -2 \\ 2 & 5 \end{bmatrix}.$$

If possible, compute:

(a) AB

(b) $AC + B^2$

(c) $AB + C^2$

Solution:

(a) $AB = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & 4 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 2 & 1 \\ 3 & 2 \end{bmatrix} = \begin{bmatrix} 14 & 8 \\ 16 & 9 \end{bmatrix}$

(b) Impossible.

(c) $AB + C^2 = \begin{bmatrix} 14 & 8 \\ 16 & 9 \end{bmatrix} + \begin{bmatrix} 3 & -2 \\ 2 & 5 \end{bmatrix} \begin{bmatrix} 3 & -2 \\ 2 & 5 \end{bmatrix} = \begin{bmatrix} 14 & 8 \\ 16 & 9 \end{bmatrix} + \begin{bmatrix} 5 & -16 \\ 16 & 21 \end{bmatrix} = \begin{bmatrix} 19 & -8 \\ 32 & 30 \end{bmatrix}$

PROPERTIES: Let A be an $m \times n$ matrix, and let B and C have sizes for which the indicated sums and products are defined. Then

(a) $A(BC) = (AB)C$

(b) $A(B + C) = AB + AC$

(c) $(B + C)A = BA + CA$

(d) $r(AB) = (rA)B = A(rB)$

(e) $(AB)^T = B^T A^T$

WARNINGS

1. In general, $AB \neq BA$.

EXAMPLE: Let $A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 1 \\ 2 & 0 \end{bmatrix}$. Then

$$AB = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 2 & 0 \end{bmatrix} = \begin{bmatrix} 5 & 1 \\ 2 & 0 \end{bmatrix}, \qquad BA = \begin{bmatrix} 1 & 1 \\ 2 & 0 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}$$

So, $AB \neq BA$.

EXAMPLE: Let $A = \begin{bmatrix} 1 & 2 & 3 \end{bmatrix}$ and $B = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}$, then

$$AB = \begin{bmatrix} 1 \cdot 4 + 2 \cdot 5 + 3 \cdot 6 \end{bmatrix} = [32]$$

and

$$BA = \begin{bmatrix} 4 \cdot 1 & 4 \cdot 2 & 4 \cdot 3 \\ 5 \cdot 1 & 5 \cdot 2 & 5 \cdot 3 \\ 6 \cdot 1 & 6 \cdot 2 & 6 \cdot 3 \end{bmatrix} = \begin{bmatrix} 4 & 8 & 12 \\ 5 & 10 & 15 \\ 6 & 12 & 18 \end{bmatrix}$$

2. If $AB = AC$, then it is not true in general that $B = C$.

EXAMPLE: Let $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$, and $C = \begin{bmatrix} 0 & 0 \\ 0 & 2 \end{bmatrix}$. Then

$$AB = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \qquad AC = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

So, $AB = AC$ and $B \neq C$.

3. If $AB = 0$, then it is not true in general that $A = 0$ or $B = 0$.

EXAMPLE: Let $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$. Then

$$AB = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

So, $AB = 0$, but $A \neq 0$ and $B \neq 0$.

Powers of Square Matrices

DEFINITION: Let A be any $n \times n$ matrix. Then the (nonnegative) powers of A are given by $A^0 = I_n$, $A^1 = A$, and for $k \geq 2$,

$$A^k = (A^{k-1})A$$

THEOREM: If A is a square matrix, and if s and t are nonnegative integers, then

(1) $A^{s+t} = A^s A^t$

(2) $(A^s)^t = A^{st} = (A^t)^s$

EXAMPLE: Let

$$A = \begin{bmatrix} 7 & 4 \\ -3 & -1 \end{bmatrix}, \qquad D = \begin{bmatrix} 1 & 0 \\ 0 & 5 \end{bmatrix}$$

then

$$D^2 = \begin{bmatrix} 1 & 0 \\ 0 & 5 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 5 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 5^2 \end{bmatrix}$$

$$D^3 = D^2 D = \begin{bmatrix} 1 & 0 \\ 0 & 5^2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 5 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 5^3 \end{bmatrix}$$

$$\cdots$$

$$D^k = D^{k-1} D = \begin{bmatrix} 1^{k-1} & 0 \\ 0 & 5^{k-1} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 5 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 5^k \end{bmatrix}$$

and

$$A^2 = \begin{bmatrix} 7 & 4 \\ -3 & -1 \end{bmatrix} \begin{bmatrix} 7 & 4 \\ -3 & -1 \end{bmatrix} = \begin{bmatrix} 37 & 24 \\ -18 & -11 \end{bmatrix}$$

$$A^3 = A^2 A = \begin{bmatrix} 37 & 24 \\ -18 & -11 \end{bmatrix} \begin{bmatrix} 7 & 4 \\ -3 & -1 \end{bmatrix} = \begin{bmatrix} 187 & 124 \\ -93 & -61 \end{bmatrix}$$

$$A^4 = A^3 A = \begin{bmatrix} 187 & 124 \\ -93 & -61 \end{bmatrix} \begin{bmatrix} 7 & 4 \\ -3 & -1 \end{bmatrix} = \begin{bmatrix} 937 & 624 \\ -468 & -311 \end{bmatrix}$$

$$A^5 = A^4 A = \begin{bmatrix} 937 & 624 \\ -468 & -311 \end{bmatrix} \begin{bmatrix} 7 & 4 \\ -3 & -1 \end{bmatrix} = \begin{bmatrix} 4687 & 3124 \\ -2343 & -1561 \end{bmatrix}$$