Fundamental Operations with Matrices

DEFINITION: An $m \times n$ **matrix** is a rectangular array of real numbers, arranged in m rows and n columns. The elements of a matrix are called the **entries**. The expression $m \times n$ denotes the **size** of the matrix.

EXAMPLE:

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}, \qquad B = \begin{bmatrix} -3 & 1 \\ 4 & 7 \\ 8 & -5 \end{bmatrix}, \qquad C = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \end{bmatrix}$$

SPECIAL TYPES OF MATRICES:

1. A **square matrix** is an $n \times n$ matrix; that is, a matrix having the same number of rows as columns. For example, the following matrices are square:

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}, \qquad B = \begin{bmatrix} -3 & 1 & 1 \\ 4 & 7 & 1 \\ 8 & -5 & 1 \end{bmatrix}, \qquad C = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 2 & -1 & 0 & 0 \\ 1 & 0 & -2 & 1 \end{bmatrix}$$

2. A **diagonal matrix** is a square matrix in which all entries that are not on the main diagonal are zero. That is, D is diagonal if and only if it is square and $d_{ij} = 0$ for $i \neq j$. For example, the following are diagonal matrices:

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix}, \qquad B = \begin{bmatrix} -3 & 0 & 0 \\ 0 & 7 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \qquad C = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 6 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

However, the following matrices

$$A = \begin{bmatrix} 1 & 4 \\ 0 & 4 \end{bmatrix}, \qquad B = \begin{bmatrix} 0 & 2 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \qquad C = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 6 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

are *not* diagonal.

3. An **identity matrix** is a diagonal matrix with all main diagonal entries equal to 1. That is, an $n \times n$ matrix A is an identity matrix if and only if $a_{ij} = 0$ for $i \neq j$ and $a_{ii} = 1$ for $1 \leq i \leq n$. The $n \times n$ identity matrix is denoted by I_n. For example, the following are identity matrices:

$$I_1 = [\,1\,], \qquad I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \qquad I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \qquad I_4 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

If the size of the identity matrix is clear from the context, I alone may be used.

4. An **upper triangular matrix** is a square matrix with all entries *below* the main diagonal equal to zero. That is, an $n \times n$ matrix A is upper triangular if and only if $a_{ij} = 0$ for $i > j$. For example, the following are upper triangular:

$$U_2 = \begin{bmatrix} 1 & 4 \\ 0 & 4 \end{bmatrix}, \qquad U_3 = \begin{bmatrix} 0 & 2 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \qquad U_4 = \begin{bmatrix} 1 & -4 & 0 & 7 \\ 0 & -6 & 2 & 0 \\ 0 & 0 & -2 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Similarly, a **lower triangular matrix** is one in which all entries *above* the main diagonal equal zero; for example,

$$L_2 = \begin{bmatrix} 1 & 0 \\ -3 & 4 \end{bmatrix}, \qquad L_3 = \begin{bmatrix} 0 & 0 & 0 \\ -3 & 0 & 0 \\ 4 & 1 & 1 \end{bmatrix}, \qquad L_4 = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 3 & -6 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 2 & -2 & 0 & 0 \end{bmatrix}$$

is lower triangular. We use U_n to represent the set of all $n \times n$ upper triangular matrices and L_n to represent the set of all $n \times n$ lower triangular matrices.

5. A **zero matrix** is any matrix all of whose entries are zero. O_{mn} denotes the $m \times n$ zero matrix, and O_n denotes the $n \times n$ zero matrix. For example,

$$O_{22} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \qquad O_{32} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \qquad O_{24} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

are zero matrices. If the size of the zero matrix is clear from the context, O alone may be used.

DEFINITION: If A and B are $m \times n$ matrices, then the **sum** $A + B$ is the $m \times n$ matrix whose entries are the sums of the corresponding entries of A and B.

EXAMPLE: We have

$$\begin{bmatrix} 1 & -2 & -1 \\ -2 & 3 & -2 \end{bmatrix} + \begin{bmatrix} 0 & 1 & 2 \\ -2 & -4 & -1 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 1 \\ -4 & -1 & -3 \end{bmatrix}$$

REMARK: We can add matrices *only* of the same size.

EXAMPLE:

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} + \begin{bmatrix} 1 & 2 \\ 1 & 0 \\ 3 & 4 \end{bmatrix} = ???$$

DEFINITION: If r is a scalar and A is a matrix, then the **scalar multiple** rA is the matrix whose entries are r times the corresponding entries in A.

EXAMPLE: We have

$$(-2) \begin{bmatrix} 1 & 2 & -3 \\ -1 & 0 & -2 \end{bmatrix} = \begin{bmatrix} -2 & -4 & 6 \\ 2 & 0 & 4 \end{bmatrix}$$

PROPERTIES: Let A, B, and C be matrices of the same size, and let r and s be scalars. Then

(a) $A + B = B + A$

(b) $(A + B) + C = A + (B + C)$

(c) $r(A + B) = rA + rB$

(d) $(r + s)A = rA + sA$

(e) $r(sA) = (rs)A$

DEFINITION: Let A be an $m \times n$ matrix. The **transpose** of A is the $n \times m$ matrix, denoted by A^T, whose columns are formed from the corresponding rows of A.

EXAMPLE: We have

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \qquad A^T = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}$$

$$B = \begin{bmatrix} -3 & 1 \\ 4 & 7 \\ 8 & -5 \end{bmatrix} \qquad B^T = \begin{bmatrix} -3 & 4 & 8 \\ 1 & 7 & -5 \end{bmatrix}$$

$$C = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \end{bmatrix} \qquad C^T = \begin{bmatrix} 1 & 5 \\ 2 & 6 \\ 3 & 7 \\ 4 & 8 \end{bmatrix}$$

PROPERTIES: Let A and B denote matrices whose sizes are appropriate for the following sums and products. Then

(a) $(A^T)^T = A$

(b) $(A + B)^T = A^T + B^T$

(c) $(rA)^T = rA^T$ for any scalar r

DEFINITION: A matrix A is **symmetric** if and only if $A = A^T$. A matrix A is **skew-symmetric** if and only if $A = -A^T$.

EXAMPLE: Consider the following matrices:

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 6 & 7 & 8 \\ 3 & 7 & 7 & 8 \\ 4 & 8 & 8 & 8 \end{bmatrix} \qquad \text{and} \qquad B = \begin{bmatrix} 0 & 2 & -3 & 4 \\ -2 & 0 & 7 & 8 \\ 3 & -7 & 0 & -8 \\ -4 & -8 & 8 & 0 \end{bmatrix}$$

One can see that A is symmetric and B is skew-symmetric.

THEOREM: Every square matrix A can be decomposed uniquely as the sum of two matrices S and V, where S is symmetric and V is skew-symmetric. Moreover,

$$S = \frac{1}{2}(A + A^T) \qquad \text{and} \qquad V = \frac{1}{2}(A - A^T)$$