The Dot Product

Definition Let $\mathbf{x} = [x_1, x_2, \ldots, x_n]$ and $\mathbf{y} = [y_1, y_2, \ldots, y_n]$ be two vectors in $\mathbb{R}^n$. The **dot (inner) product** of $\mathbf{x}$ and $\mathbf{y}$ is given by

$$\mathbf{x} \cdot \mathbf{y} = x_1y_1 + x_2y_2 + \cdots + x_ny_n = \sum_{k=1}^{n} x_ky_k.$$

EXAMPLE: Let $\mathbf{x} = [2, -5, -1]$ and $\mathbf{y} = [3, 2, -3]$, then

$$\mathbf{x} \cdot \mathbf{y} = 2 \cdot 3 + (-5) \cdot 2 + (-1)(-3) = -1$$

Theorem 1.5 If $\mathbf{x}$, $\mathbf{y}$, and $\mathbf{z}$ are any vectors in $\mathbb{R}^n$, and if c is any scalar, then

(1) $\mathbf{x} \cdot \mathbf{y} = \mathbf{y} \cdot \mathbf{x}$	Commutativity of Dot Product
(2) $\mathbf{x} \cdot \mathbf{x} = \|\mathbf{x}\|^2 \geq 0$	Relationship between Dot Product and Length
(3) $\mathbf{x} \cdot \mathbf{x} = 0$ if and only if $\mathbf{x} = \mathbf{0}$	
(4) $c(\mathbf{x} \cdot \mathbf{y}) = (c\mathbf{x}) \cdot \mathbf{y} = \mathbf{x} \cdot (c\mathbf{y})$	Relationship between Scalar Multiplication and Dot Product
(5) $\mathbf{x} \cdot (\mathbf{y} + \mathbf{z}) = (\mathbf{x} \cdot \mathbf{y}) + (\mathbf{x} \cdot \mathbf{z})$	Distributive Laws of Dot Product
(6) $(\mathbf{x} + \mathbf{y}) \cdot \mathbf{z} = (\mathbf{x} \cdot \mathbf{z}) + (\mathbf{y} \cdot \mathbf{z})$	over Addition

Proof: Let $\mathbf{x} = [x_1, x_2, \ldots, x_n]$, $\mathbf{y} = [y_1, y_2, \ldots, y_n]$, and $\mathbf{z} = [z_1, z_2, \ldots, z_n]$. Then,

(1) $\mathbf{x} \cdot \mathbf{y} = x_1y_1 + x_2y_2 + \ldots + x_ny_n = y_1x_1 + y_2x_2 + \ldots + y_nx_n = \mathbf{y} \cdot \mathbf{x}$

(2) $\mathbf{x} \cdot \mathbf{x} = x_1x_1 + x_2x_2 + \ldots + x_nx_n = x_1^2 + x_2^2 + \ldots + x_n^2 = \left(\sqrt{x_1^2 + x_2^2 + \ldots + x_n^2}\right)^2 = \|\mathbf{x}\|^2 \geq 0$

(3) Since $\mathbf{x} \cdot \mathbf{x} = x_1^2 + x_2^2 + \ldots + x_n^2$, it follows that $\mathbf{x} \cdot \mathbf{x} = 0$ if and only if $x_1^2 + x_2^2 + \ldots + x_n^2 = 0$ which holds only when $x_1 = x_2 = \ldots = x_n = 0$, that is, $\mathbf{x} = \mathbf{0}$.

(4) We have

$$\begin{aligned} c(\mathbf{x} \cdot \mathbf{y}) &= c(x_1y_1 + x_2y_2 + \ldots + x_ny_n) \\ &= (cx_1)y_1 + (cx_2)y_2 + \ldots + (cx_n)y_n = (c\mathbf{x}) \cdot \mathbf{y} \end{aligned}$$

In the same way we prove that $c(\mathbf{x} \cdot \mathbf{y}) = \mathbf{x} \cdot (c\mathbf{y})$.

(5) We have

$$\begin{aligned} \mathbf{x} \cdot (\mathbf{y} + \mathbf{z}) &= [x_1, x_2, \ldots, x_n] \cdot ([y_1, y_2, \ldots, y_n] + [z_1, z_2, \ldots, z_n]) \\ &= [x_1, x_2, \ldots, x_n] \cdot [y_1 + z_1, y_2 + z_2, \ldots, y_n + z_n] \\ &= x_1(y_1 + z_1) + x_2(y_2 + z_2) + \ldots + x_n(y_n + z_n) \\ &= x_1y_1 + x_1z_1 + x_2y_2 + x_2z_2 + \ldots + x_ny_n + x_nz_n \\ &= (x_1y_1 + x_2y_2 + \ldots + x_ny_n) + (x_1z_1 + x_2z_2 + \ldots + x_nz_n) \\ &= [x_1, x_2, \ldots, x_n] \cdot [y_1, y_2, \ldots, y_n] + [x_1, x_2, \ldots, x_n] \cdot [z_1, z_2, \ldots, z_n] \\ &= (\mathbf{x} \cdot \mathbf{y}) + (\mathbf{x} \cdot \mathbf{z}) \end{aligned}$$

In the same way we prove that $(\mathbf{x} + \mathbf{y}) \cdot \mathbf{z} = (\mathbf{x} \cdot \mathbf{z}) + (\mathbf{y} \cdot \mathbf{z})$.

EXAMPLE: We have

$$\begin{aligned}(5\mathbf{x}-4\mathbf{y})\cdot(-2\mathbf{x}+3\mathbf{y}) &= [(5\mathbf{x}-4\mathbf{y})\cdot(-2\mathbf{x})]+[(5\mathbf{x}-4\mathbf{y})\cdot(3\mathbf{y})]\\ &= [(5\mathbf{x})\cdot(-2\mathbf{x})]+[(-4\mathbf{y})\cdot(-2\mathbf{x})]+[(5\mathbf{x})\cdot(3\mathbf{y})]+[(-4\mathbf{y})\cdot(3\mathbf{y})]\\ &= -10(\mathbf{x}\cdot\mathbf{x})+8(\mathbf{y}\cdot\mathbf{x})+15(\mathbf{x}\cdot\mathbf{y})-12(\mathbf{y}\cdot\mathbf{y})\\ &= -10\|\mathbf{x}\|^2+23(\mathbf{x}\cdot\mathbf{y})-12\|\mathbf{y}\|^2\end{aligned}$$

Inequalities Involving the Dot Product

Theorem 1.6 (Cauchy-Schwarz Inequality) If $\mathbf{x}$ and $\mathbf{y}$ are vectors in $\mathbb{R}^n$, then $|\mathbf{x}\cdot\mathbf{y}| \leq (\|\mathbf{x}\|)(\|\mathbf{y}\|)$.

REMARK: Let $\mathbf{x}=[x_1,x_2,\ldots,x_n]$ and $\mathbf{y}=[y_1,y_2,\ldots,y_n]$. Then the Cauchy-Schwarz Inequality becomes

$$|x_1y_1+x_2y_2+\ldots+x_ny_n| \leq \sqrt{x_1^2+x_2^2+\ldots+x_n^2}\sqrt{y_1^2+y_2^2+\ldots+y_n^2}$$

Proof: If either $\mathbf{x}=\mathbf{0}$ or $\mathbf{y}=\mathbf{0}$, the theorem is certainly true. Hence, we need only examine the case when both $\|\mathbf{x}\|$ and $\|\mathbf{y}\|$ are nonzero. We need to prove

$$-(\|\mathbf{x}\|)(\|\mathbf{y}\|) \leq \mathbf{x}\cdot\mathbf{y} \leq (\|\mathbf{x}\|)(\|\mathbf{y}\|)$$

This statement is true if and only if

$$-1 \leq \frac{\mathbf{x}\cdot\mathbf{y}}{(\|\mathbf{x}\|)(\|\mathbf{y}\|)} \leq 1$$

which can be rewritten as

$$-1 \leq \frac{\mathbf{x}}{\|\mathbf{x}\|}\cdot\frac{\mathbf{y}}{\|\mathbf{y}\|} \leq 1$$

Note that $\dfrac{\mathbf{x}}{\|\mathbf{x}\|}$ and $\dfrac{\mathbf{y}}{\|\mathbf{y}\|}$ are both *unit* vectors. Thus, it is enough to show that

$$-1 \leq \mathbf{a}\cdot\mathbf{b} \leq 1$$

for any unit vectors $\mathbf{a}$ and $\mathbf{b}$. The term $\mathbf{a}\cdot\mathbf{b}$ occurs as part of the expansion of $(\mathbf{a}+\mathbf{b})\cdot(\mathbf{a}+\mathbf{b})$, as well as part of $(\mathbf{a}-\mathbf{b})\cdot(\mathbf{a}-\mathbf{b})$. We have

$$\begin{aligned}(\mathbf{a}+\mathbf{b})\cdot(\mathbf{a}+\mathbf{b}) = \|\mathbf{a}+\mathbf{b}\|^2 &\geq 0\\ (\mathbf{a}\cdot\mathbf{a})+(\mathbf{a}\cdot\mathbf{b})+(\mathbf{b}\cdot\mathbf{a})+(\mathbf{b}\cdot\mathbf{b}) &\geq 0\\ (\mathbf{a}\cdot\mathbf{a})+2(\mathbf{a}\cdot\mathbf{b})+(\mathbf{b}\cdot\mathbf{b}) &\geq 0\\ \|\mathbf{a}\|^2+2(\mathbf{a}\cdot\mathbf{b})+\|\mathbf{b}\|^2 &\geq 0\\ 1+2(\mathbf{a}\cdot\mathbf{b})+1 &\geq 0\\ 2(\mathbf{a}\cdot\mathbf{b}) &\geq -2\\ (\mathbf{a}\cdot\mathbf{b}) &\geq -1\end{aligned}$$

and

$$\begin{aligned}(\mathbf{a}-\mathbf{b})\cdot(\mathbf{a}-\mathbf{b}) = \|\mathbf{a}-\mathbf{b}\|^2 &\geq 0\\ (\mathbf{a}\cdot\mathbf{a})-(\mathbf{a}\cdot\mathbf{b})-(\mathbf{b}\cdot\mathbf{a})+(\mathbf{b}\cdot\mathbf{b}) &\geq 0\\ (\mathbf{a}\cdot\mathbf{a})-2(\mathbf{a}\cdot\mathbf{b})+(\mathbf{b}\cdot\mathbf{b}) &\geq 0\\ \|\mathbf{a}\|^2-2(\mathbf{a}\cdot\mathbf{b})+\|\mathbf{b}\|^2 &\geq 0\\ 1-2(\mathbf{a}\cdot\mathbf{b})+1 &\geq 0\\ -2(\mathbf{a}\cdot\mathbf{b}) &\geq -2\\ (\mathbf{a}\cdot\mathbf{b}) &\leq 1\end{aligned}$$

Hence, $-1 \leq \mathbf{a}\cdot\mathbf{b} \leq 1$. ■

REMARK: For more information, see Appendix I.

Theorem 1.7 (Triangle Inequality) If $\mathbf{x}$ and $\mathbf{y}$ are vectors in $\mathbb{R}^n$, then $\|\mathbf{x}+\mathbf{y}\| \leq \|\mathbf{x}\| + \|\mathbf{y}\|$.

REMARK: Let $\mathbf{x} = [x_1, x_2, \ldots, x_n]$ and $\mathbf{y} = [y_1, y_2, \ldots, y_n]$. Then

$$\mathbf{x}+\mathbf{y} = [x_1 + y_1,\ x_2 + y_2, \ldots, x_n + y_n]$$

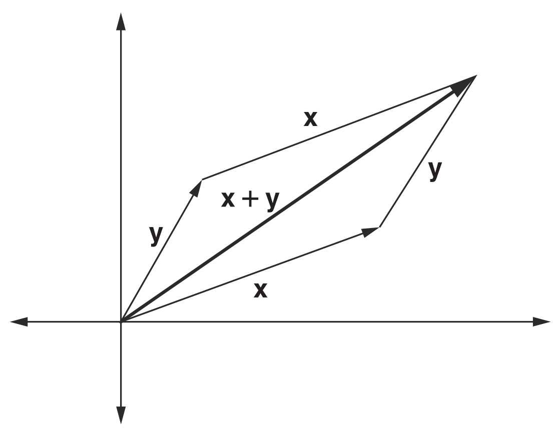
hence

$$\|\mathbf{x}+\mathbf{y}\| = \sqrt{(x_1+y_1)^2 + (x_2+y_2)^2 + \ldots + (x_n+y_n)^2}$$

therefore the Triangle Inequality becomes

$$\sqrt{(x_1+y_1)^2 + (x_2+y_2)^2 + \ldots + (x_n+y_n)^2} \leq \sqrt{x_1^2 + x_2^2 + \ldots + x_n^2} + \sqrt{y_1^2 + y_2^2 + \ldots + y_n^2}$$

We can prove the theorem above geometrically in $\mathbb{R}^2$ and $\mathbb{R}^3$ by noting that the length of $\mathbf{x}+\mathbf{y}$, one side of the triangles in the Figure below, is never larger than the sum of the lengths of the other two sides, $\mathbf{x}$ and $\mathbf{y}$. For more information, see Appendix II.



Proof: The following algebraic proof extends the above result to $\mathbb{R}^n$ for $n > 3$. We have

$$\begin{aligned}
\|\mathbf{x}+\mathbf{y}\|^2 &= (\mathbf{x}+\mathbf{y}) \cdot (\mathbf{x}+\mathbf{y}) \\
&= (\mathbf{x}\cdot\mathbf{x}) + (\mathbf{x}\cdot\mathbf{y}) + (\mathbf{y}\cdot\mathbf{x}) + (\mathbf{y}\cdot\mathbf{y}) \\
&= (\mathbf{x}\cdot\mathbf{x}) + 2(\mathbf{x}\cdot\mathbf{y}) + (\mathbf{y}\cdot\mathbf{y}) \\
&= \|\mathbf{x}\|^2 + 2(\mathbf{x}\cdot\mathbf{y}) + \|\mathbf{y}\|^2 \\
&\leq \|\mathbf{x}\|^2 + 2|\mathbf{x}\cdot\mathbf{y}| + \|\mathbf{y}\|^2
\end{aligned}$$

Since $|\mathbf{x}\cdot\mathbf{y}| \leq \|\mathbf{x}\|\|\mathbf{y}\|$ by the Cauchy-Schwarz Inequality, this implies

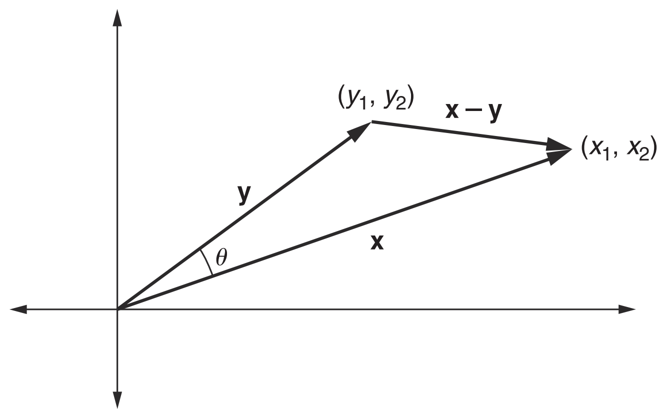
$$\begin{aligned}
\|\mathbf{x}+\mathbf{y}\|^2 &\leq \|\mathbf{x}\|^2 + 2\|\mathbf{x}\|\|\mathbf{y}\| + \|\mathbf{y}\|^2 \\
&= (\|\mathbf{x}\| + \|\mathbf{y}\|)^2
\end{aligned}$$

Taking the square root of both sides, we get the desired result. ■

The Angle between Two Vectors

The dot product enables us to find the angle θ between two nonzero vectors $\mathbf{x}$ and $\mathbf{y}$ in $\mathbb{R}^2$ or $\mathbb{R}^3$ that begin at the same initial point. There are actually *two* angles formed by the vectors $\mathbf{x}$ and $\mathbf{y}$, but we always choose the angle θ between two vectors to be the one measuring between 0 and π radians.

Consider the vector $\mathbf{x} - \mathbf{y}$ in the Figure below, which begins at the terminal point of $\mathbf{y}$ and ends at the terminal point of $\mathbf{x}$.



Because $0 \leq \theta \leq \pi$, it follows from the Law of Cosines (see Appendix III) that

$$\|\mathbf{x} - \mathbf{y}\|^2 = \|\mathbf{x}\|^2 + \|\mathbf{y}\|^2 - 2(\|\mathbf{x}\|)(\|\mathbf{y}\|)\cos\theta$$

On the other hand,

$$\begin{aligned}\|\mathbf{x} - \mathbf{y}\|^2 &= (\mathbf{x} - \mathbf{y}) \cdot (\mathbf{x} - \mathbf{y}) \\ &= (\mathbf{x} \cdot \mathbf{x}) - (\mathbf{x} \cdot \mathbf{y}) - (\mathbf{y} \cdot \mathbf{x}) + (\mathbf{y} \cdot \mathbf{y}) \\ &= (\mathbf{x} \cdot \mathbf{x}) - 2(\mathbf{x} \cdot \mathbf{y}) + (\mathbf{y} \cdot \mathbf{y}) \\ &= \|\mathbf{x}\|^2 - 2(\mathbf{x} \cdot \mathbf{y}) + \|\mathbf{y}\|^2\end{aligned}$$

Hence,

$$\|\mathbf{x}\|^2 + \|\mathbf{y}\|^2 - 2(\|\mathbf{x}\|)(\|\mathbf{y}\|)\cos\theta = \|\mathbf{x}\|^2 - 2(\mathbf{x} \cdot \mathbf{y}) + \|\mathbf{y}\|^2$$

therefore

$$-2(\|\mathbf{x}\|)(\|\mathbf{y}\|)\cos\theta = -2(\mathbf{x} \cdot \mathbf{y})$$

Dividing both sides by $-2(\|\mathbf{x}\|)(\|\mathbf{y}\|)$ yields

$$\cos\theta = \frac{\mathbf{x} \cdot \mathbf{y}}{(\|\mathbf{x}\|)(\|\mathbf{y}\|)}$$

EXAMPLE: Suppose $\mathbf{x} = [6, -4]$ and $\mathbf{y} = [-2, 3]$ and θ is the angle between $\mathbf{x}$ and $\mathbf{y}$. Then,

$$\begin{aligned}\cos\theta = \frac{\mathbf{x} \cdot \mathbf{y}}{(\|\mathbf{x}\|)(\|\mathbf{y}\|)} &= \frac{(6)(-2) + (-4)(3)}{\sqrt{6^2 + (-4)^2}\sqrt{(-2)^2 + 3^2}} = \frac{(-12) + (-12)}{\sqrt{52}\sqrt{13}} \\ &= \frac{-24}{\sqrt{4}\sqrt{13}\sqrt{13}} = -\frac{12}{13} \approx -0.9231\end{aligned}$$

Using a calculator, we find that $\theta \approx 2.75$ radians, or 157.4°. (Remember that $0 \leq \theta \leq \pi$.)

In higher-dimensional spaces, we are outside the geometry of everyday experience, and in such cases, we have not yet defined the angle between two vectors. However, by the Cauchy-Schwarz Inequality, $(\mathbf{x} \cdot \mathbf{y})/(\|\mathbf{x}\|\|\mathbf{y}\|)$ always has a value between -1 and 1 for any nonzero vectors $\mathbf{x}$ and $\mathbf{y}$ in $\mathbb{R}^n$. Thus, this value equals $\cos\theta$ for a unique θ between 0 and π radians. Hence, we can define the angle between two vectors in $\mathbb{R}^n$ so it is consistent with the situation in $\mathbb{R}^2$ and $\mathbb{R}^3$.

Definition Let $\mathbf{x}$ and $\mathbf{y}$ be two nonzero vectors in $\mathbb{R}^n$, for $n \geq 2$. Then the **angle between x and y** is the unique angle between 0 and π radians whose cosine is $(\mathbf{x} \cdot \mathbf{y})/((\|\mathbf{x}\|)(\|\mathbf{y}\|))$.

EXAMPLE: Suppose $\mathbf{x} = [-1, 4, 2, 0, -3]$ and $\mathbf{y} = [2, 1, -4, -1, 0]$. Then,

$$\frac{\mathbf{x} \cdot \mathbf{y}}{(\|\mathbf{x}\|)(\|\mathbf{y}\|)} = -\frac{6}{2\sqrt{165}} \approx -0.234$$

Using a calculator, we find the angle θ between $\mathbf{x}$ and $\mathbf{y}$ is approximately 1.8 radians, or 103.5°.

The following theorem is an immediate consequence of the last definition:

Theorem 1.8 Let $\mathbf{x}$ and $\mathbf{y}$ be nonzero vectors in $\mathbb{R}^n$, and let θ be the angle between $\mathbf{x}$ and $\mathbf{y}$. Then,

(1) $\mathbf{x} \cdot \mathbf{y} > 0$ if and only if $0 \leq \theta < \frac{\pi}{2}$ radians ($0°$ or *acute*).

(2) $\mathbf{x} \cdot \mathbf{y} = 0$ if and only if $\theta = \frac{\pi}{2}$ radians ($90°$).

(3) $\mathbf{x} \cdot \mathbf{y} < 0$ if and only if $\frac{\pi}{2} < \theta \leq \pi$ radians ($180°$ or *obtuse*).

Special Cases: Orthogonal and Parallel Vectors

Definition Two vectors $\mathbf{x}$ and $\mathbf{y}$ in $\mathbb{R}^n$ are **orthogonal (perpendicular)** if and only if $\mathbf{x} \cdot \mathbf{y} = 0$.

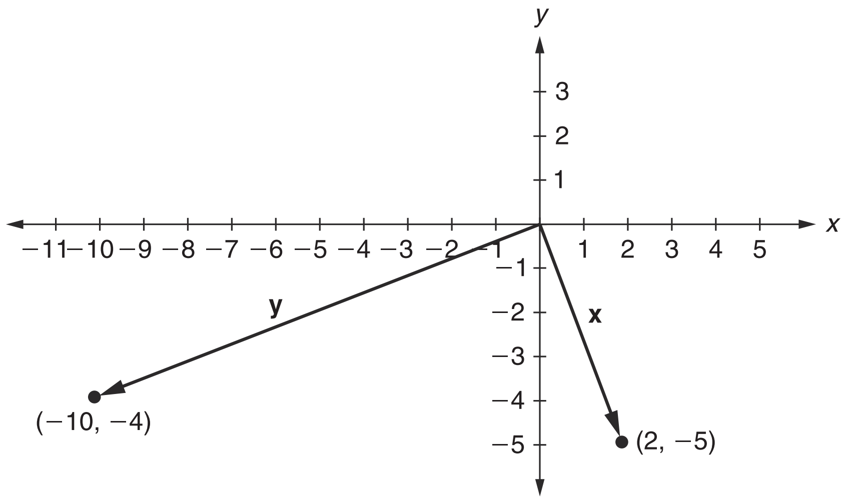
Theorem 1.9 Let $\mathbf{x}$ and $\mathbf{y}$ be nonzero vectors in $\mathbb{R}^n$. Then $\mathbf{x}$ and $\mathbf{y}$ are parallel if and only if $\mathbf{x} \cdot \mathbf{y} = \pm\|\mathbf{x}\|\|\mathbf{y}\|$ (that is, $\cos\theta = \pm 1$, where θ is the angle between $\mathbf{x}$ and $\mathbf{y}$).

EXAMPLE: The vectors $\mathbf{x} = [5, -3, 1]$ and $\mathbf{y} = [6, 9, -3]$ are orthogonal, since

$$\cos\theta = \frac{\mathbf{x} \cdot \mathbf{y}}{(\|\mathbf{x}\|)(\|\mathbf{y}\|)} = \frac{(5)(6) + (-3)(9) + (1)(-3)}{\sqrt{5^2 + (-3)^2 + 1^2}\sqrt{6^2 + 9^2 + (-3)^2}} = \frac{0}{\sqrt{35}\sqrt{126}} = 0$$

EXAMPLE: The vectors $\mathbf{x} = [2, -5]$ and $\mathbf{y} = [-10, -4]$ are orthogonal, since

$$\cos\theta = \frac{\mathbf{x}\cdot\mathbf{y}}{(\|\mathbf{x}\|)(\|\mathbf{y}\|)} = \frac{(2)(-10)+(-5)(-4)}{\sqrt{2^2+(-5)^2}\sqrt{(-10)^2+(-4)^2}} = \frac{(-20)+(20)}{\sqrt{29}\sqrt{116}} = 0$$



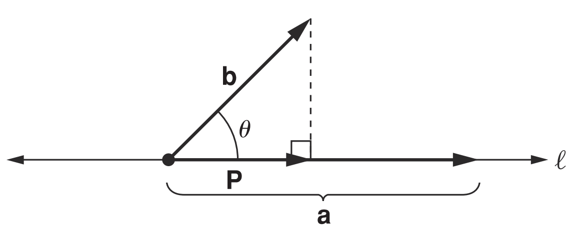
Projection Vectors

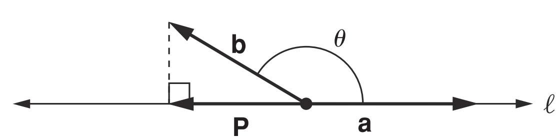
By the projection $\mathbf{p}$ of $\mathbf{b}$ onto $\mathbf{a}$, we mean the vector from the initial point of $\mathbf{a}$ to the point where the dropped perpendicular meets the line ℓ.

Definition If $\mathbf{a}$ and $\mathbf{b}$ are vectors in $\mathbb{R}^n$, with $\mathbf{a} \neq \mathbf{0}$, then the **projection vector of b onto a** is

$$\mathbf{proj_a b} = \left(\frac{\mathbf{a}\cdot\mathbf{b}}{\|\mathbf{a}\|^2}\right)\mathbf{a}.$$

Indeed, using trigonometry, we see that when $0 \le \theta \le \pi/2$, the vector $\mathbf{p}$ has length $\|\mathbf{b}\|\cos\theta$ and is in the direction of the unit vector $\mathbf{a}/\|\mathbf{a}\|$. Also, when $\pi/2 < \theta \le \pi$, $\mathbf{p}$ has length $-\|\mathbf{b}\|\cos\theta$ and is in the direction of the unit vector $-\mathbf{a}/\|\mathbf{a}\|$.





Therefore, we can express $\mathbf{p}$ in all cases as

$$\mathbf{p} = (\|\mathbf{b}\|\cos\theta)\left(\frac{\mathbf{a}}{\|\mathbf{a}\|}\right)$$

But we know that

$$\cos\theta = \frac{\mathbf{a}\cdot\mathbf{b}}{(\|\mathbf{a}\|)(\|\mathbf{b}\|)}$$

and hence

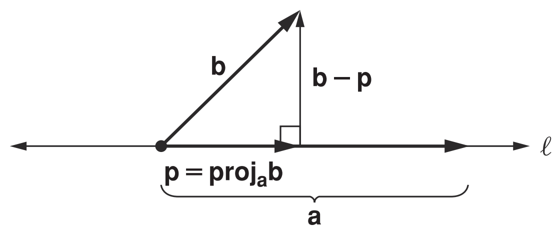
$$\mathbf{p} = \left(\|\mathbf{b}\|\frac{\mathbf{a}\cdot\mathbf{b}}{(\|\mathbf{a}\|)(\|\mathbf{b}\|)}\right)\left(\frac{\mathbf{a}}{\|\mathbf{a}\|}\right) = \left(\frac{\mathbf{a}\cdot\mathbf{b}}{\|\mathbf{a}\|^2}\right)\mathbf{a}$$

EXAMPLE: Let $\mathbf{a} = [4, 2]$ and $\mathbf{b} = [7, 6]$. Find the orthogonal projection of $\mathbf{b}$ onto $\mathbf{a}$.

Solution: We have

$$\mathbf{proj_a b} = \left(\frac{\mathbf{a} \cdot \mathbf{b}}{\|\mathbf{a}\|^2}\right)\mathbf{a} = \frac{4 \cdot 7 + 2 \cdot 6}{4 \cdot 4 + 2 \cdot 2}\mathbf{a} = 2[4, 2] = [8, 4]$$

The projection vector can be used to decompose a given vector $\mathbf{b}$ into the sum of two **component vectors**. Suppose $\mathbf{a} \neq \mathbf{0}$. Notice that if $\mathbf{proj_a b} \neq \mathbf{0}$, then it is parallel to $\mathbf{a}$ by definition because it is a scalar multiple of $\mathbf{a}$.



Also, $\mathbf{b} - \mathbf{proj_a b}$ is orthogonal to $\mathbf{a}$ because

$$\begin{aligned}(\mathbf{b} - \mathbf{proj_a b}) \cdot \mathbf{a} &= \mathbf{b} \cdot \mathbf{a} - (\mathbf{proj_a b}) \cdot \mathbf{a} \\ &= \mathbf{b} \cdot \mathbf{a} - \left(\frac{\mathbf{a} \cdot \mathbf{b}}{\|\mathbf{a}\|^2}\right)(\mathbf{a} \cdot \mathbf{a}) \\ &= \mathbf{b} \cdot \mathbf{a} - \left(\frac{\mathbf{a} \cdot \mathbf{b}}{\|\mathbf{a}\|^2}\right)\|\mathbf{a}\|^2 \\ &= 0\end{aligned}$$

Because

$$\mathbf{proj_a b} + (\mathbf{b} - \mathbf{proj_a b}) = \mathbf{b}$$

we have proved

> **Theorem 1.10** Let $\mathbf{a}$ be a nonzero vector in $\mathbb{R}^n$, and let $\mathbf{b}$ be any vector in $\mathbb{R}^n$. Then $\mathbf{b}$ can be decomposed as the sum of two component vectors, $\mathbf{proj_a b}$ and $\mathbf{b} - \mathbf{proj_a b}$, where the first (if nonzero) is parallel to $\mathbf{a}$ and the second is orthogonal to $\mathbf{a}$.

EXAMPLE: Consider $\mathbf{a} = [4, 2]$ and $\mathbf{b} = [7, 6]$ from the Example above, where we found the component of $\mathbf{b}$ in the direction of the vector $\mathbf{a}$ is $[8, 4]$. Then the component of $\mathbf{b}$ orthogonal to $\mathbf{a}$ is

$$\mathbf{b} - \mathbf{proj_a b} = [7, 6] - [8, 4] = [-1, 2]$$

We can easily check that $\mathbf{b} - \mathbf{proj_a b}$ is orthogonal to $\mathbf{a}$ as follows:

$$(\mathbf{b} - \mathbf{proj_a b}) \cdot \mathbf{a} = (4)(-1) + (2)(2) = 0$$

Appendix I

If $n = 2$, the Cauchy-Schwarz Inequality becomes $|x_1y_1 + x_2y_2| \leq \sqrt{x_1^2 + x_2^2}\sqrt{y_1^2 + y_2^2}$. We have

$$|x_1y_1 + x_2y_2| \leq \sqrt{x_1^2 + x_2^2}\sqrt{y_1^2 + y_2^2}$$

$$\Uparrow$$

$$\sqrt{(x_1y_1 + x_2y_2)^2} \leq \sqrt{(x_1^2 + x_2^2)(y_1^2 + y_2^2)}$$

$$\Uparrow$$

$$(x_1y_1 + x_2y_2)^2 \leq (x_1^2 + x_2^2)(y_1^2 + y_2^2)$$

$$\Uparrow$$

$$x_1^2y_1^2 + 2x_1y_1x_2y_2 + x_2^2y_2^2 \leq x_1^2y_1^2 + x_1^2y_2^2 + x_2^2y_1^2 + x_2^2y_2^2$$

$$\Uparrow$$

$$2x_1y_1x_2y_2 \leq x_1^2y_2^2 + x_2^2y_1^2$$

$$\Uparrow$$

$$0 \leq x_1^2y_2^2 - 2x_1y_1x_2y_2 + x_2^2y_1^2$$

$$\Uparrow$$

$$0 \leq (x_1y_2 - x_2y_1)^2 = \begin{vmatrix} x_1 & x_2 \\ y_1 & y_2 \end{vmatrix}^2$$

where

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

is a determinant.

If $n = 3$, the Cauchy-Schwarz Inequality becomes

$$|x_1y_1 + x_2y_2 + x_3y_3| \le \sqrt{x_1^2 + x_2^2 + x_3^2}\sqrt{y_1^2 + y_2^2 + y_3^2}$$

We have

$$\begin{aligned}
|x_1y_1 + x_2y_2 + x_3y_3| &\le \sqrt{x_1^2 + x_2^2 + x_3^2}\sqrt{y_1^2 + y_2^2 + y_3^2} \\
&\Uparrow \\
\sqrt{(x_1y_1 + x_2y_2 + x_3y_3)^2} &\le \sqrt{(x_1^2 + x_2^2 + x_3^2)(y_1^2 + y_2^2 + y_3^2)} \\
&\Uparrow \\
(x_1y_1 + x_2y_2 + x_3y_3)^2 &\le (x_1^2 + x_2^2 + x_3^2)(y_1^2 + y_2^2 + y_3^2) \\
&\Uparrow \\
x_1^2y_1^2 + x_2^2y_2^2 + x_3^2y_3^2 + 2x_1y_1x_2y_2 + 2x_1y_1x_3y_3 + 2x_2y_2x_3y_3 &\le x_1^2y_1^2 + x_1^2y_2^2 + x_1^2y_3^2 + x_2^2y_1^2 + x_2^2y_2^2 + x_2^2y_3^2 \\
&\quad + x_3^2y_1^2 + x_3^2y_2^2 + x_3^2y_3^2 \\
&\Uparrow \\
2x_1y_1x_2y_2 + 2x_1y_1x_3y_3 + 2x_2y_2x_3y_3 &\le x_1^2y_2^2 + x_1^2y_3^2 + x_2^2y_1^2 + x_2^2y_3^2 + x_3^2y_1^2 + x_3^2y_2^2
\end{aligned}$$

which can be rewritten as

$$\begin{aligned}
0 &\le x_1^2y_2^2 - 2x_1y_1x_2y_2 + x_2^2y_1^2 + x_1^2y_3^2 - 2x_1y_1x_3y_3 + x_3^2y_1^2 + x_2^2y_3^2 - 2x_2y_2x_3y_3 + x_3^2y_2^2 \\
&\Uparrow \\
0 &\le (x_1y_2 - x_2y_1)^2 + (x_1y_3 - x_3y_1)^2 + (x_2y_3 - x_3y_2)^2 = \begin{vmatrix} x_1 & x_2 \\ y_1 & y_2 \end{vmatrix}^2 + \begin{vmatrix} x_1 & x_3 \\ y_1 & y_3 \end{vmatrix}^2 + \begin{vmatrix} x_2 & x_3 \\ y_2 & y_3 \end{vmatrix}^2
\end{aligned}$$

This suggests the following general approach: We have

$$\sum_{\substack{(i,j)=(1,2) \\ i<j}}^{(n-1,n)} \begin{vmatrix} x_i & x_j \\ y_i & y_j \end{vmatrix}^2 \ge 0$$

This can be rewritten (should be proven rigorously!) as

$$\left(\sum_{i=1}^{n} x_iy_i\right)^2 \le \left(\sum_{i=1}^{n} x_i^2\right)\left(\sum_{i=1}^{n} y_i^2\right)$$

Taking the square root of both sides, we get

$$\left|\sum_{i=1}^{n} x_iy_i\right| \le \sqrt{\sum_{i=1}^{n} x_i^2}\sqrt{\sum_{i=1}^{n} y_i^2}$$

which gives us the desired result.

Appendix II

If $n = 2$, the Triangle Inequality becomes $\sqrt{(x_1+y_1)^2+(x_2+y_2)^2} \le \sqrt{x_1^2+x_2^2} + \sqrt{y_1^2+y_2^2}$. We have

$$\sqrt{(x_1+y_1)^2+(x_2+y_2)^2} \le \sqrt{x_1^2+x_2^2} + \sqrt{y_1^2+y_2^2}$$

$$\Uparrow$$

$$\left(\sqrt{(x_1+y_1)^2+(x_2+y_2)^2}\right)^2 \le \left(\sqrt{x_1^2+x_2^2} + \sqrt{y_1^2+y_2^2}\right)^2$$

$$\Uparrow$$

$$(x_1+y_1)^2+(x_2+y_2)^2 \le \left(\sqrt{x_1^2+x_2^2}\right)^2 + 2\sqrt{x_1^2+x_2^2}\sqrt{y_1^2+y_2^2} + \left(\sqrt{y_1^2+y_2^2}\right)^2$$

$$\Uparrow$$

$$x_1^2+2x_1y_1+y_1^2+x_2^2+2x_2y_2+y_2^2 \le x_1^2+x_2^2+2\sqrt{x_1^2+x_2^2}\sqrt{y_1^2+y_2^2}+y_1^2+y_2^2$$

$$\Uparrow$$

$$2x_1y_1+2x_2y_2 \le 2\sqrt{x_1^2+x_2^2}\sqrt{y_1^2+y_2^2}$$

$$\Uparrow$$

$$x_1y_1+x_2y_2 \le \sqrt{x_1^2+x_2^2}\sqrt{y_1^2+y_2^2}$$

$$\Uparrow$$

$$(x_1y_1+x_2y_2)^2 \le (x_1^2+x_2^2)(y_1^2+y_2^2)$$

$$\Uparrow$$

$$x_1^2y_1^2+2x_1y_1x_2y_2+x_2^2y_2^2 \le x_1^2y_1^2+x_1^2y_2^2+x_2^2y_1^2+x_2^2y_2^2$$

$$\Uparrow$$

$$2x_1y_1x_2y_2 \le x_1^2y_2^2+x_2^2y_1^2$$

$$\Uparrow$$

$$0 \le x_1^2y_2^2-2x_1y_1x_2y_2+x_2^2y_1^2$$

$$\Uparrow$$

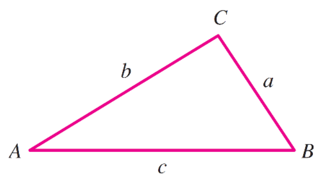
$$0 \le (x_1y_2-x_2y_1)^2 = \begin{vmatrix} x_1 & x_2 \\ y_1 & y_2 \end{vmatrix}^2$$

where

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

is a determinant.

Appendix III



The Law of Cosines

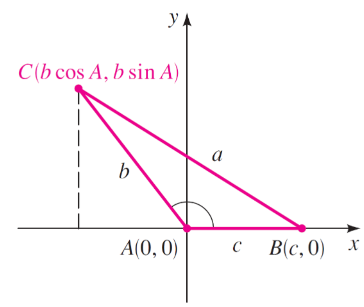
In any triangle *ABC* (see Figure 1), we have

$$a^2 = b^2 + c^2 - 2bc\cos A$$

$$b^2 = a^2 + c^2 - 2ac\cos B$$

$$c^2 = a^2 + b^2 - 2ab\cos C$$

Proof: To prove the Law of Cosines, place triangle ABC so that $\angle A$ is at the origin, as shown in the Figure on the right. The coordinates of the vertices B and C are $(c, 0)$ and $(b\cos A, b\sin A)$, respectively. Using the Distance Formula, we get



$$\begin{aligned} a^2 &= (c - b\cos A)^2 + (b\sin A - 0)^2 \\ &= c^2 - 2bc\cos A + b^2\cos^2 A + b^2\sin^2 A \\ &= c^2 - 2bc\cos A + b^2(\cos^2 A + \sin^2 A) \\ &= b^2 + c^2 - 2bc\cos A \end{aligned}$$

This proves the first formula. The other two formulas are obtained in the same way by placing each of the other vertices of the triangle at the origin and repeating the preceding argument. ■