Fundamental Operations with Vectors

Definition A **real n-vector** is an ordered sequence of n real numbers (sometimes referred to as an **ordered n-tuple** of real numbers). The set of all n-vectors is denoted $\mathbb{R}^n$.

For example, $\mathbb{R}^2$ is the set of all 2-vectors (ordered 2-tuples = ordered pairs) of real numbers; it includes $[2, -4]$ and $[-6.2, 3.14]$. $\mathbb{R}^3$ is the set of all 3-vectors (ordered 3-tuples = ordered triples) of real numbers; it includes $[2, -3, 0]$ and $[-\sqrt{2}, 42.7, \pi]$. $\mathbb{R}^4$ is the set of all 4-vectors (ordered 4-tuples = ordered quadruples) of real numbers; it includes $[-1, -7, 0, 5]$ and $[0.5, -19.1, 0, 100]$.

The vector in $\mathbb{R}^n$ that has all n entries equal to zero is called the **zero n-vector**. In $\mathbb{R}^2$ and $\mathbb{R}^3$, the zero vectors are $[0, 0]$ and $[0, 0, 0]$, respectively.

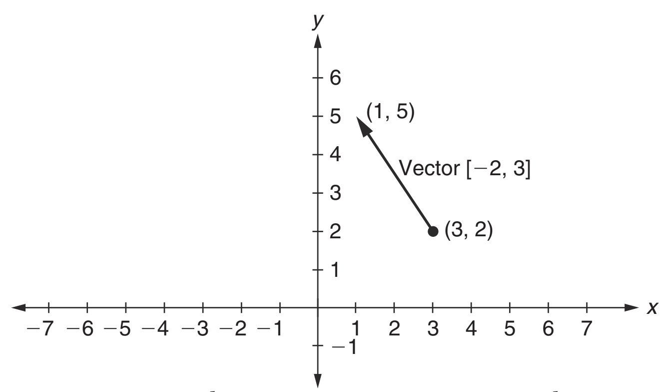
Two vectors in $\mathbb{R}^n$ are **equal** if and only if all corresponding entries (called **coordinates**) in their n-tuples agree. That is, $[x_1, x_2, \ldots, x_n] = [y_1, y_2, \ldots, y_n]$ if and only if $x_1 = y_1,\ x_2 = y_2, \ldots,$ and $x_n = y_n$.

A single number (such as -10 or 2.6) is often called a **scalar** to distinguish it from a vector.

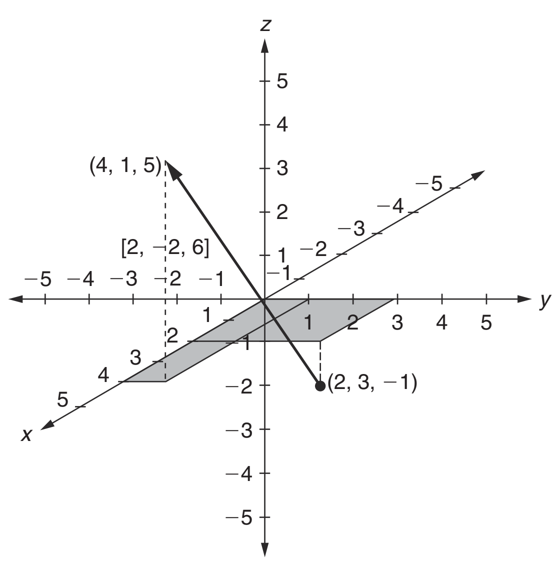
Geometric Interpretation of Vectors

Vectors in $\mathbb{R}^2$ frequently represent movement from one point to another in a coordinate plane. From initial point $(3, 2)$ to terminal point $(1, 5)$, there is a net decrease of 2 units along the x-axis and a net increase of 3 units along the y-axis. A vector representing this change would thus be $[-2, 3]$.

Vectors can be positioned at any desired starting point. For example, $[-2, 3]$ could also represent a movement from initial point $(9, -6)$ to terminal point $(7, -3)$.



Vectors in $\mathbb{R}^3$ have a similar geometric interpretation: a 3-vector is used to represent movement between points in three-dimensional space. For example, $[2, -2, 6]$ can represent movement from initial point $(2, 3, -1)$ to terminal point $(4, 1, 5)$.



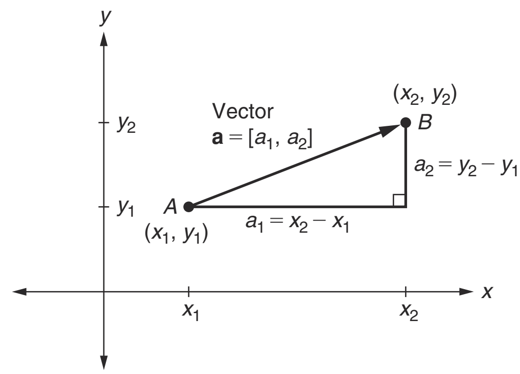
Length of a Vector

Recall the **distance formula** in the plane; the distance between two points (x_1, y_1) and (x_2, y_2) is

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

This formula arises from the Pythagorean Theorem for right triangles. The 2-vector between the points is $[a_1, a_2]$, where $a_1 = x_2 - x_1$ and $a_2 = y_2 - y_1$, so

$$d = \sqrt{a_1^2 + a_2^2}$$



Definition The **length** (also known as the **norm** or **magnitude**) of a vector $\mathbf{a} = [a_1, a_2, \ldots, a_n]$ in $\mathbb{R}^n$ is $\|\mathbf{a}\| = \sqrt{a_1^2 + a_2^2 + \cdots + a_n^2}$.

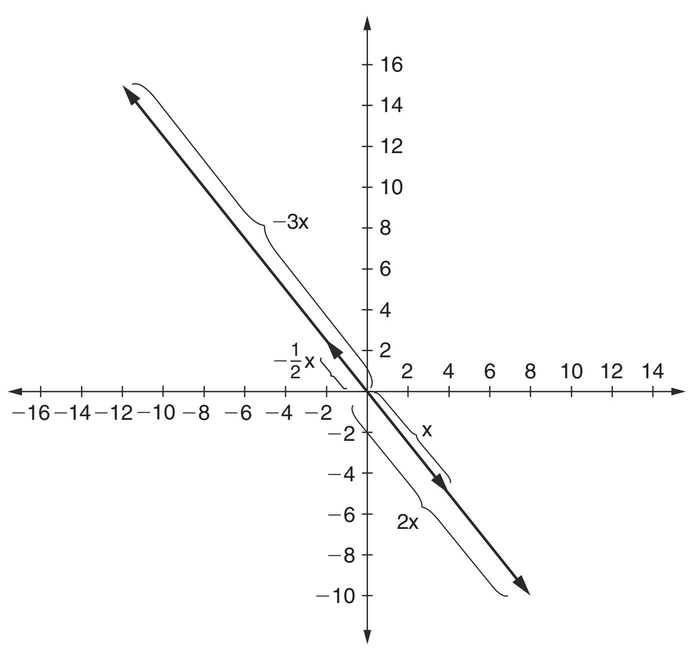
EXAMPLE: The length of the vector $\mathbf{a} = [4, -3, 0, 2]$ is given by

$$\|\mathbf{a}\| = \sqrt{4^2 + (-3)^2 + 0^2 + 2^2} = \sqrt{16 + 9 + 4} = \sqrt{29}$$

Definition Any vector of length 1 is called a **unit vector**.

Scalar Multiplication and Parallel Vectors

Definition Let $\mathbf{x} = [x_1, x_2, \ldots, x_n]$ be a vector in $\mathbb{R}^n$, and let c be any scalar (real number). Then $c\mathbf{x}$, the **scalar multiple of x by** c, is the vector $[cx_1, cx_2, \ldots, cx_n]$.



Theorem 1.1 Let $\mathbf{x} \in \mathbb{R}^n$, and let c be any real number (scalar). Then $\|c\mathbf{x}\| = |c|\ \|\mathbf{x}\|$. That is, the length of $c\mathbf{x}$ is the absolute value of c times the length of $\mathbf{x}$.

Proof: Let $\mathbf{x} = [x_1, x_2, \ldots, x_n]$, then $c\mathbf{x} = [cx_1, cx_2, \ldots, cx_n]$. Therefore

$$\|c\mathbf{x}\| = \sqrt{(cx_1)^2 + \ldots + (cx_n)^2} = \sqrt{c^2(x_1^2 + \ldots + x_n^2)}$$

$$= \sqrt{c^2}\sqrt{x_1^2 + \ldots + x_n^2)} = |c|\sqrt{x_1^2 + \ldots + x_n^2} = |c|\|\mathbf{x}\| \quad \blacksquare$$

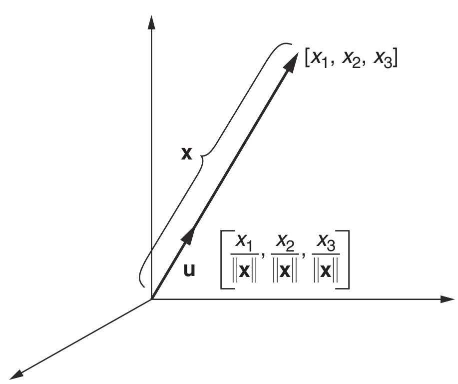
Definition Two nonzero vectors $\mathbf{x}$ and $\mathbf{y}$ in $\mathbb{R}^n$ are **in the same direction** if and only if there is a positive real number c such that $\mathbf{y} = c\mathbf{x}$. Two nonzero vectors $\mathbf{x}$ and $\mathbf{y}$ are **in opposite directions** if and only if there is a negative real number c such that $\mathbf{y} = c\mathbf{x}$. Two nonzero vectors are **parallel** if and only if they are either in the same direction or in the opposite direction.

Corollary 1.2 If $\mathbf{x}$ is a nonzero vector in $\mathbb{R}^n$, then $\mathbf{u} = (1/\|\mathbf{x}\|)\mathbf{x}$ is a unit vector in the same direction as $\mathbf{x}$.

Proof: Since $1/\|\mathbf{x}\|$ is positive, $\mathbf{x}$ and $\mathbf{u}$ have the same direction. We have

$$\|\mathbf{u}\| = \left\| \frac{1}{\|\mathbf{x}\|}\mathbf{x} \right\| = \frac{1}{\|\mathbf{x}\|}\|\mathbf{x}\| = 1$$

Hence $\mathbf{u}$ is a unit vector. $\blacksquare$



REMARK: The process of "dividing" a vector by its length to obtain a unit vector in the same direction is called **normalizing** the vector.

EXAMPLE: Consider the vector $[2, 3, -1, 1]$ in $\mathbb{R}^4$. Because

$$\|[2, 3, -1, 1]\| = \sqrt{15}$$

normalizing $[2, 3, -1, 1]$ gives a unit vector $\mathbf{u}$ in the same direction as $[2, 3, -1, 1]$, which is

$$\mathbf{u} = \left(\frac{1}{\sqrt{15}}\right)[2, 3, -1, 1] = \left[\frac{2}{\sqrt{15}}, \frac{3}{\sqrt{15}}, \frac{-1}{\sqrt{15}}, \frac{1}{\sqrt{15}}\right]$$

Addition and Subtraction with Vectors

> **Definition** Let $\mathbf{x} = [x_1, x_2, \ldots, x_n]$ and $\mathbf{y} = [y_1, y_2, \ldots, y_n]$ be vectors in $\mathbb{R}^n$. Then $\mathbf{x} + \mathbf{y}$, the **sum** of $\mathbf{x}$ and $\mathbf{y}$, is the vector $[x_1 + y_1, x_2 + y_2, \ldots, x_n + y_n]$ in $\mathbb{R}^n$.

Vectors are added by summing their respective coordinates. For example, if

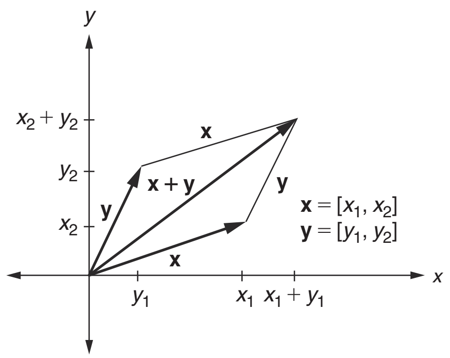
$$\mathbf{x} = [2, -3, 5] \qquad \text{and} \qquad \mathbf{y} = [-6, 4, -2]$$

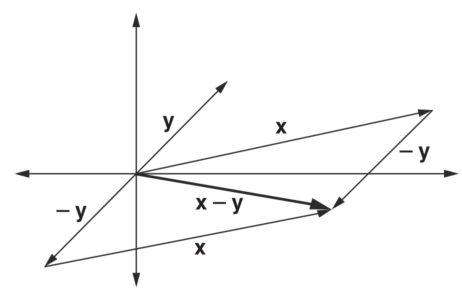
then

$$\mathbf{x} + \mathbf{y} = [2 + (-6),\ -3 + 4,\ 5 + (-2)] = [-4, 1, 3]$$

Vectors cannot be added unless they have the same number of coordinates.

Let $-\mathbf{y}$ denote the scalar multiple $-1\mathbf{y}$. We can now define **subtraction** of vectors in a natural way: if $\mathbf{x}$ and $\mathbf{y}$ are both vectors in $\mathbb{R}^n$, let $\mathbf{x} - \mathbf{y}$ be the vector $\mathbf{x} + (-\mathbf{y})$.





Fundamental Properties of Addition and Scalar Multiplication

> **Theorem 1.3** Let $\mathbf{x} = [x_1, x_2, \ldots, x_n]$, $\mathbf{y} = [y_1, y_2, \ldots, y_n]$, and $\mathbf{z} = [z_1, z_2, \ldots, z_n]$ be any vectors in $\mathbb{R}^n$, and let c and d be any real numbers (scalars). Let $\mathbf{0}$ represent the zero vector in $\mathbb{R}^n$. Then
>
> (1) $\mathbf{x} + \mathbf{y} = \mathbf{y} + \mathbf{x}$ — Commutative Law of Addition
> (2) $\mathbf{x} + (\mathbf{y} + \mathbf{z}) = (\mathbf{x} + \mathbf{y}) + \mathbf{z}$ — Associative Law of Addition
> (3) $\mathbf{0} + \mathbf{x} = \mathbf{x} + \mathbf{0} = \mathbf{x}$ — Existence of Identity Element for Addition
> (4) $\mathbf{x} + (-\mathbf{x}) = (-\mathbf{x}) + \mathbf{x} = \mathbf{0}$ — Existence of Inverse Elements for Addition
> (5) $c(\mathbf{x} + \mathbf{y}) = c\mathbf{x} + c\mathbf{y}$ — Distributive Laws of Scalar Multiplication
> (6) $(c + d)\mathbf{x} = c\mathbf{x} + d\mathbf{x}$ — over Addition
> (7) $(cd)\mathbf{x} = c(d\mathbf{x})$ — Associativity of Scalar Multiplication
> (8) $1\mathbf{x} = \mathbf{x}$ — Identity Property for Scalar Multiplication

> **Theorem 1.4** Let $\mathbf{x}$ be a vector in $\mathbb{R}^n$, and let c be a scalar. If $c\mathbf{x} = \mathbf{0}$, then either $c = 0$ or $\mathbf{x} = \mathbf{0}$.

Linear Combinations of Vectors

> **Definition** Let $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$ be vectors in $\mathbb{R}^n$. Then the vector $\mathbf{v}$ is a **linear combination** of $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$ if and only if there are scalars $c_1, c_2, \ldots, c_k$ such that $\mathbf{v} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_k\mathbf{v}_k$.

Thus, a linear combination of vectors is a sum of scalar multiples of those vectors. For example,the vector $[-2, 8, 5, 0]$ is a linear combination of

$$[3, 1, -2, 2], \quad [1, 0, 3, -1], \quad \text{and} \quad [4, -2, 1, 0]$$

because

$$2[3, 1, -2, 2] + 4[1, 0, 3, -1] + (-3)[4, -2, 1, 0] = [-2, 8, 5, 0]$$

Note that any vector in $\mathbb{R}^3$ can be expressed in a unique way as a linear combination of

$$\mathbf{i} = [1, 0, 0], \quad \mathbf{j} = [0, 1, 0], \quad \text{and} \quad \mathbf{k} = [0, 0, 1]$$

For example,

$$[3, -2, 5] = 3[1, 0, 0] + (-2)[0, 1, 0] + 5[0, 0, 1] = 3\mathbf{i} - 2\mathbf{j} + 5\mathbf{k}$$

In general,

$$[a, b, c] = a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$$

Also, every vector in $\mathbb{R}^n$ can be expressed as a linear combination of the standard unit vectors

$$\mathbf{e_1} = [1, 0, 0, \ldots, 0], \quad \mathbf{e_2} = [0, 1, 0, \ldots, 0], \quad \ldots, \quad \mathbf{e_n} = [0, 0, 0, \ldots, 1]$$

since

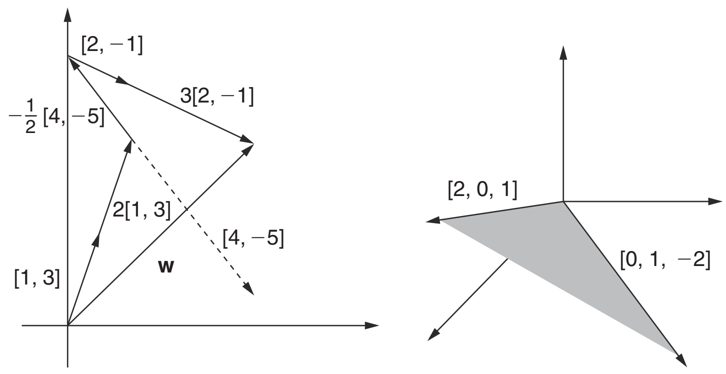
$$\begin{aligned}[a_1, a_2, \ldots, a_n] &= [a_1, 0, 0, \ldots, 0] + [0, a_2, 0, \ldots, 0] + \ldots + [0, 0, 0, \ldots, a_n] \\ &= a_1[1, 0, 0, \ldots, 0] + a_2[0, 1, 0, \ldots, 0] + \ldots + a_n[0, 0, 0, \ldots, 1] \\ &= a_1\mathbf{e_1} + a_2\mathbf{e_2} + \ldots + a_n\mathbf{e_n}\end{aligned}$$

One helpful way to picture linear combinations of the vectors $\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_k}$ is to remember that each vector represents a certain amount of movement in a particular direction. When we combine these vectors using addition and scalar multiplication, the endpoint of each linear combination vector represents a "destination" that can be reached using these operations. For example, the linear combination

$$\mathbf{w} = 2[1, 3] - \frac{1}{2}[4, -5] + 3[2, -1] = \left[6, \frac{11}{2}\right]$$

is the destination reached by traveling in the direction of $[1, 3]$, but traveling *twice* its length, then traveling in the direction opposite to $[4, -5]$, but *half* its length, and finally traveling in the direction $[2, -1]$, but *three times* its length (see Figure (a) below).

We can also consider the set of all possible destinations that can be reached using linear combinations of a certain set of vectors. For example, the set of all linear combinations in $\mathbb{R}^3$ of $\mathbf{v_1} = [2, 0, 1]$ and $\mathbf{v_2} = [0, 1, -2]$ is the set of all vectors (beginning at the origin) with endpoints lying in the plane through the origin containing $\mathbf{v_1}$ and $\mathbf{v_2}$ (see Figure (b) below).

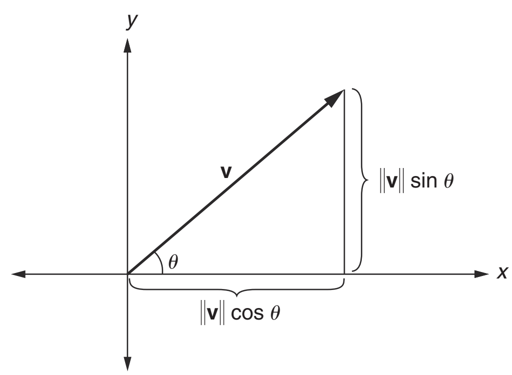


(a) (b)

Physical Applications of Addition and Scalar Multiplication

Addition and scalar multiplication of vectors are often used to solve problems in elementary physics. Recall the trigonometric fact that if $\mathbf{v}$ is a vector in $\mathbb{R}^2$ forming an angle of θ with the positive x-axis, then

$$\mathbf{v} = [\|\mathbf{v}\| \cos\theta,\ \|\mathbf{v}\| \sin\theta]$$



EXAMPLE: Suppose a man swims 5 km/hr in calm water. If he is swimming toward the east in a wide stream with a northwest current of 3 km/hr, what is his **resultant velocity** (net speed and direction)?

Solution: The velocities of the swimmer and current are shown as vectors in the Figure below, where we have, for convenience, placed the swimmer at the origin. Now,

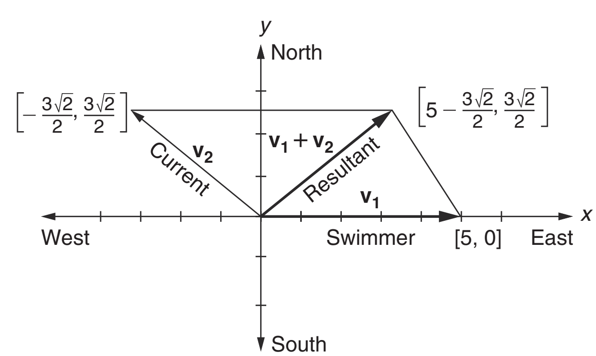
$$\mathbf{v_1} = [5, 0] \quad \text{and} \quad \mathbf{v_2} = [3\cos 135°,\ 3\sin 135°] = \left[-3\frac{\sqrt{2}}{2},\ 3\frac{\sqrt{2}}{2}\right]$$

Thus, the total (resultant) velocity of the swimmer is the sum of these velocities, $\mathbf{v_1} + \mathbf{v_2}$, which is

$$\left[5 - 3\frac{\sqrt{2}}{2},\ 3\frac{\sqrt{2}}{2}\right] \approx [2.88, 2.12]$$

Hence, each hour the swimmer is traveling about 2.9 km east and 2.1 km north. The resultant speed of the swimmer is

$$\left\| \left[5 - 3\frac{\sqrt{2}}{2},\ 3\frac{\sqrt{2}}{2}\right] \right\| \approx 3.58 \text{ km/hr}$$



EXAMPLE (**Newton's Second Law**): Newton's famous Second Law of Motion asserts that the sum, $\mathbf{f}$, of the vector forces on an object is equal to the scalar multiple of the mass m of the object times the vector acceleration $\mathbf{a}$ of the object; that is,

$$\mathbf{f} = m\mathbf{a}$$

For example, suppose a mass of 5 kg (kilograms) in a three-dimensional coordinate system has two forces acting on it: a force $\mathbf{f_1}$ of 10 newtons in the direction of the vector $[-2, 1, 2]$ and a force $\mathbf{f_2}$ of 20 newtons in the direction of the vector $[6, 3, -2]$. What is the acceleration of the object?

Solution: We must first normalize the direction vectors $[-2, 1, 2]$ and $[6, 3, -2]$ so that their lengths do not contribute to the magnitude of the forces $\mathbf{f_1}$ and $\mathbf{f_2}$. Therefore,

$$\mathbf{f_1} = 10\frac{[-2, 1, 2]}{\|[-2, 1, 2]\|} \qquad \text{and} \qquad \mathbf{f_2} = 20\frac{[6, 3, -2]}{\|[6, 3, -2]\|}$$

The net force on the object is

$$\mathbf{f} = \mathbf{f_1} + \mathbf{f_2}$$

Thus, the net acceleration on the object is

$$\mathbf{a} = \frac{1}{m}\mathbf{f} = \frac{1}{m}(\mathbf{f_1} + \mathbf{f_2}) = \frac{1}{5}\left(10\frac{[-2, 1, 2]}{\|[-2, 1, 2]\|} + 20\frac{[6, 3, -2]}{\|[6, 3, -2]\|}\right)$$

which equals

$$\frac{2}{3}[-2, 1, 2] + \frac{4}{7}[6, 3, -2] = \left[\frac{44}{21}, \frac{50}{21}, \frac{4}{21}\right]$$

The length of $\mathbf{a}$ is approximately 3.18, so pulling out a factor of 3.18 from each coordinate, we can approximate $\mathbf{a}$ as $3.18[0.66, 0.75, 0.06]$, where $[0.66, 0.75, 0.06]$ is a *unit* vector. Hence, the acceleration is about 3.18 m/sec^2 in the direction $[0.66, 0.75, 0.06]$.