Inner Product Spaces

DEFINITION: An **inner product** on a vector space V is a function that, to each pair of vectors $\mathbf{u}$ and $\mathbf{v}$ in V, associates a real number $\langle \mathbf{u}, \mathbf{v} \rangle$ and satisfies the following axioms, for all $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$ in $\mathbf{V}$ and all scalars c:

1. $\langle \mathbf{u}, \mathbf{v} \rangle = \langle \mathbf{v}, \mathbf{u} \rangle$

2. $\langle \mathbf{u} + \mathbf{v}, \mathbf{w} \rangle = \langle \mathbf{u}, \mathbf{w} \rangle + \langle \mathbf{v}, \mathbf{w} \rangle$

3. $\langle c\mathbf{u}, \mathbf{v} \rangle = c\langle \mathbf{u}, \mathbf{v} \rangle$

4. $\langle \mathbf{u}, \mathbf{u} \rangle \geq 0$ and $\langle \mathbf{u}, \mathbf{u} \rangle = 0$ if and only if $\mathbf{u} = \mathbf{0}$

A vector space with an inner product is called an **inner product space**.

EXAMPLE 1:
(a) Fix any two positive numbers — say, 4 and 5 — and for vectors $\mathbf{u} = (u_1, u_2)$ and $\mathbf{v} = (v_1, v_2)$ in $\mathbb{R}^2$, set

$$\langle \mathbf{u}, \mathbf{v} \rangle = 4u_1v_1 + 5u_2v_2 \tag{1}$$

Show that equation (1) defines an inner product.

Solution: Certainly Axiom 1 is satisfied, because

$$\langle \mathbf{u}, \mathbf{v} \rangle = 4u_1v_1 + 5u_2v_2 = 4v_1u_1 + 5v_2u_2 = \langle \mathbf{v}, \mathbf{u} \rangle$$

If $\mathbf{w} = (w_1, w_2)$, then

$$\begin{aligned}
\langle \mathbf{u} + \mathbf{v}, \mathbf{w} \rangle &= 4(u_1 + v_1)w_1 + 5(u_2 + v_2)w_2 \\
&= 4u_1w_1 + 4v_1w_1 + 5u_2w_2 + 5v_2w_2 \\
&= 4u_1w_1 + 5u_2w_2 + 4v_1w_1 + 5v_2w_2 \\
&= \langle \mathbf{u}, \mathbf{w} \rangle + \langle \mathbf{v}, \mathbf{w} \rangle
\end{aligned}$$

This verifies Axiom 2. For Axiom 3, compute

$$\langle c\mathbf{u}, \mathbf{v} \rangle = 4(cu_1)v_1 + 5(cu_2)v_2 = c(4u_1v_1 + 5u_2v_2) = c\langle \mathbf{u}, \mathbf{v} \rangle$$

For Axiom 4, note that

$$\langle \mathbf{u}, \mathbf{u} \rangle = 4u_1^2 + 5u_2^2 \geq 0$$

and

$$4u_1^2 + 5u_2^2 = 0$$

only if $u_1 = u_2 = 0$, that is, if $\mathbf{u} = \mathbf{0}$. Also, $\langle \mathbf{0}, \mathbf{0} \rangle = 0$. So (1) defines an inner product on $\mathbb{R}^2$.

(b) Let $d_1, \ldots, d_n$ be some positive numbers. For vectors $\mathbf{u} = (u_1, \ldots, u_n)$ and $\mathbf{v} = (v_1, \ldots, v_n)$ in $\mathbb{R}^n$, set

$$\langle \mathbf{u}, \mathbf{v} \rangle = d_1u_1v_1 + \ldots + d_nu_nv_n \tag{2}$$

Show that equation (1) defines an inner product.

Solution: Axiom 1 is satisfied, because

$$\langle \mathbf{u}, \mathbf{v} \rangle = d_1u_1v_1 + \ldots + d_nu_nv_n = d_1v_1u_1 + \ldots + d_nv_nu_n = \langle \mathbf{v}, \mathbf{u} \rangle$$

If $\mathbf{w} = (w_1, \ldots, w_n)$, then

$$\begin{aligned}\langle \mathbf{u}+\mathbf{v}, \mathbf{w}\rangle &= d_1(u_1+v_1)w_1 + \ldots + d_n(u_n+v_n)w_n \\ &= d_1u_1w_1 + d_1v_1w_1 + \ldots + d_nu_nw_n + d_nv_nw_n \\ &= d_1u_1w_1 + \ldots + d_nu_nw_n + d_1v_1w_1 + \ldots + d_nv_nw_n \\ &= \langle \mathbf{u}, \mathbf{w}\rangle + \langle \mathbf{v}, \mathbf{w}\rangle\end{aligned}$$

This verifies Axiom 2. For Axiom 3, compute

$$\langle c\mathbf{u}, \mathbf{v}\rangle = d_1(cu_1)v_1 + \ldots + d_n(cu_n)v_n = c(d_1u_1v_1 + \ldots + d_nu_nv_n) = c\langle \mathbf{u}, \mathbf{v}\rangle$$

For Axiom 4, note that

$$\langle \mathbf{u}, \mathbf{u}\rangle = d_1u_1^2 + \ldots + d_nu_2^2 \geq 0$$

and

$$d_1u_1^2 + \ldots + d_nu_2^2 = 0$$

only if $u_1 = \ldots = u_2 = 0$, that is, if $\mathbf{u} = \mathbf{0}$. Also, $\langle \mathbf{0}, \mathbf{0}\rangle = 0$. So (2) defines an inner product on $\mathbb{R}^n$.

EXAMPLE 2:

(a) For

$$p(x) = p_nt^n + \ldots + p_1t + p_0 \qquad \text{and} \qquad q(x) = q_nt^n + \ldots + q_1t + q_0$$

in $\mathbb{P}_n$, define

$$\langle p, q\rangle = d_np_nq_n + \ldots + d_1p_1q_1 + d_0p_0q_0 \tag{3}$$

where $d_1, \ldots, d_n$ are some positive numbers. Axiom 1 is satisfied, because

$$\begin{aligned}\langle p, q\rangle &= d_np_nq_n + \ldots + d_1p_1q_1 + d_0p_0q_0 \\ &= d_nq_np_n + \ldots + d_1q_1p_1 + d_0q_0p_0 = \langle q, p\rangle\end{aligned}$$

If r is in $\mathbb{P}_2$, then

$$\begin{aligned}\langle p+q, r\rangle &= d_n(p_n+q_n)r_n + \ldots + d_1(p_1+q_1)r_1 + d_0(p_0+q_0)r_0 \\ &= d_np_nr_n + d_nq_nr_n + \ldots + d_1p_1r_1 + d_1q_1r_1 + d_0p_0r_0 + d_0q_0r_0 \\ &= d_np_nr_n + \ldots + d_1p_1r_1 + d_0p_0r_0 + d_nq_nr_n + \ldots + d_1q_1r_1 + d_0q_0r_0 \\ &= \langle p, r\rangle + \langle q, r\rangle\end{aligned}$$

This verifies Axiom 2. For Axiom 3, compute

$$\langle cp, q\rangle = d_n(cp_n)q_n + \ldots + d_1(cp_1)q_1 + d_0(cp_0)q_0 = c(d_np_nq_n + \ldots + d_1p_1q_1 + d_0p_0q_0) = c\langle p, q\rangle$$

For Axiom 4, note that

$$\langle p, p\rangle = d_np_n^2 + \ldots + d_1p_1^2 + d_0p_0^2 \geq 0$$

and

$$d_np_n^2 + \ldots + d_1p_1^2 + d_0p_0^2 = 0$$

only if $p_n = \ldots = p_1 = p_0 = 0$, that is, if $p = 0$. Also, $\langle \mathbf{0}, \mathbf{0}\rangle = 0$. (The boldface zero here denotes the zero polynomial, the zero vector in $\mathbb{P}_n$.) So (3) defines an inner product on $\mathbb{P}_n$.

(b) Let $t_0, \ldots, t_n$ be distinct real numbers. For p and q in $\mathbb{P}_n$, define

$$\langle p, q\rangle = p(t_0)q(t_0) + p(t_1)q(t_1) + \ldots + p(t_n)q(t_n) \tag{4}$$

Axiom 1 is satisfied, because

$$\begin{aligned}\langle p, q\rangle &= p(t_0)q(t_0) + p(t_1)q(t_1) + \ldots + p(t_n)q(t_n)\\ &= q(t_0)p(t_0) + q(t_1)p(t_1) + \ldots + q(t_n)p(t_n) = \langle q, p\rangle\end{aligned}$$

If r is in $\mathbb{P}_2$, then

$$\begin{aligned}\langle p+q, r\rangle &= \Big(p(t_0)+q(t_0)\Big)r(t_0) + \Big(p(t_1)+q(t_1)\Big)r(t_1) + \ldots + \Big(p(t_n)+q(t_n)\Big)r(t_n)\\ &= p(t_0)r(t_0) + q(t_0)r(t_0) + p(t_1)r(t_1) + q(t_1)r(t_1) + \ldots + p(t_n)r(t_n) + q(t_n)r(t_n)\\ &= p(t_0)r(t_0) + p(t_1)r(t_1) + \ldots + p(t_n)r(t_n) + q(t_0)r(t_0) + q(t_1)r(t_1) + \ldots + q(t_n)r(t_n)\\ &= \langle p, r\rangle + \langle q, r\rangle\end{aligned}$$

This verifies Axiom 2. For Axiom 3, compute

$$\begin{aligned}\langle cp, q\rangle &= cp(t_0)q(t_0) + cp(t_1)q(t_1) + \ldots + cp(t_n)q(t_n)\\ &= c\Big(p(t_0)q(t_0) + p(t_1)q(t_1) + \ldots + p(t_n)q(t_n)\Big) = c\langle p, q\rangle\end{aligned}$$

For Axiom 4, note that

$$\langle p, p\rangle = [p(t_0)]^2 + [p(t_1)]^2 + \ldots + [p(t_n)]^2 \geq 0$$

Also, $\langle \mathbf{0}, \mathbf{0}\rangle = 0$. If $\langle p, p\rangle = 0$, then p must vanish at $n+1$ points: $t_0, \ldots, t_n$. This is possible only if p is the zero polynomial, because the degree of p is less than $n+1$. Thus (4) defines an inner product on $\mathbb{P}_n$.

EXAMPLE 3: Let V be $\mathbb{P}_2$, with the inner product from Example 2b, where

$$t_0 = 0, \quad t_1 = \frac{1}{2}, \quad \text{and} \quad t_2 = 1$$

Let $p(t) = 12t^2$ and $q(t) = 2t - 1$. Compute $\langle p, q\rangle$ and $\langle q, q\rangle$.

Solution: We have

$$\begin{aligned}\langle p, q\rangle &= p(0)q(0) + p\left(\frac{1}{2}\right)q\left(\frac{1}{2}\right) + p(1)q(1)\\ &= (0)(-1) + (3)(0) + (12)(1) = 12\end{aligned}$$

$$\begin{aligned}\langle q, q\rangle &= [q(0)]^2 + \left[q\left(\frac{1}{2}\right)\right]^2 + [q(1)]^2\\ &= (-1)^2 + (0)^2 + (1)^2 = 2\end{aligned}$$

Lengths, Distances, and Orthogonality

Let V be an inner product space, with the inner product denoted by $\langle \mathbf{u}, \mathbf{v} \rangle$. Just as in $\mathbb{R}^n$, we define the **length**, or **norm**, of a vector $\mathbf{v}$ to be the scalar

$$\|\mathbf{v}\| = \sqrt{\langle \mathbf{v}, \mathbf{v} \rangle}$$

Equivalently, $\|\mathbf{v}\|^2 = \langle \mathbf{v}, \mathbf{v} \rangle$. (This definition makes sense because $\langle \mathbf{v}, \mathbf{v} \rangle \geq 0$, but the definition *does not* say that $\langle \mathbf{v}, \mathbf{v} \rangle$ is a "sum of squares," because $\mathbf{v}$ need not be an element of $\mathbb{R}^n$.)

A **unit vector** is one whose length is 1. The **distance between u and v** is $\|\mathbf{u} - \mathbf{v}\|$. Vectors $\mathbf{u}$ and $\mathbf{v}$ are **orthogonal** if $\langle \mathbf{u}, \mathbf{v} \rangle = 0$.

EXAMPLE 4: Let V be $\mathbb{P}_2$, with the inner product from Example 2b, where $t_0 = 0,\ t_1 = 1/2$, and $t_2 = 1$. Compute the lengths of the vectors $p(t) = 12t^2$ and $q(t) = 2t - 1$.

Solution: We have

$$\|p\|^2 = \langle p, p \rangle = [p(0)]^2 + \left[p\left(\frac{1}{2}\right)\right]^2 + [p(1)]^2 \ = 0 + [3]^2 + [12]^2 = 153$$

hence

$$\|p\| = \sqrt{153}$$

From Example 3, $\langle q, q \rangle = 2$. Hence $\|q\| = \sqrt{2}$.

The Gram-Schmidt Process

EXAMPLE 5: Let V be $\mathbb{P}_4$ with the inner product in Example 2b, involving evaluation of polynomials at $-2, -1, 0, 1$, and 2, and view $\mathbb{P}_2$ as a subspace of V. Produce an orthogonal basis for $\mathbb{P}_2$ by applying the Gram-Schmidt process to the polynomials $1, t$, and t^2.

Solution: The inner product depends only on the values of a polynomial at $-2, \ldots, 2$, so we list the values of each polynomial as a vector in $\mathbb{R}^5$, underneath the name of the polynomial:

$$\begin{array}{cccc} \text{Polynomial:} & 1 & t & t^2 \\ \text{Vector of values:} & \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, & \begin{bmatrix} -2 \\ -1 \\ 0 \\ 1 \\ 2 \end{bmatrix}, & \begin{bmatrix} 4 \\ 1 \\ 0 \\ 1 \\ 4 \end{bmatrix} \end{array}$$

The inner product of two polynomials in V equals the (standard) inner product of their corresponding vectors in $\mathbb{R}^5$. Observe that t is orthogonal to the constant function 1, since

$$\langle t, 1 \rangle = (-2) \cdot 1 + (-1) \cdot 1 + 0 \cdot 1 + 1 \cdot 1 + 2 \cdot 1 = 0$$

So take

$$p_0(t) = 1 \qquad \text{and} \qquad p_1(t) = t$$

We have

$$\langle t^2, p_0 \rangle = \langle t^2, 1 \rangle = 4 \cdot 1 + 1 \cdot 1 + 0 \cdot 1 + 1 \cdot 1 + 4 \cdot 1 = 10$$

$$\langle p_0, p_0 \rangle = \langle 1, 1 \rangle = 1 \cdot 1 + 1 \cdot 1 + 1 \cdot 1 + 1 \cdot 1 + 1 \cdot 1 = 5$$

$$\langle t^2, p_1 \rangle = \langle t^2, t \rangle = 4 \cdot (-2) + 1 \cdot (-1) + 0 \cdot 0 + 1 \cdot 1 + 4 \cdot 2 = 0$$

By the Gram-Schmidt Process,

$$p_2(t) = t^2 - \frac{\langle t^2, p_0\rangle}{\langle p_0, p_0\rangle} p_0 - \frac{\langle t^2, p_1\rangle}{\langle p_1, p_1\rangle} p_1 = t^2 - \frac{10}{5} p_0 - \frac{0}{\langle p_1, p_1\rangle} p_1 = t^2 - 2\cdot 1 - 0\cdot t = t^2 - 2$$

An orthogonal basis for the subspace $\mathbb{P}_2$ of V is:

$$\begin{array}{lccc} \text{Polynomial:} & p_0 & p_1 & p_2 \\ \text{Vector of values:} & \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, & \begin{bmatrix} -2 \\ -1 \\ 0 \\ 1 \\ 2 \end{bmatrix}, & \begin{bmatrix} 2 \\ -1 \\ -2 \\ -1 \\ 2 \end{bmatrix} \end{array} \tag{4}$$

EXAMPLE 6: Let V be $\mathbb{P}_4$ with the inner product in Example 5, and let p_0, p_1, and p_2 be the orthogonal basis found in Example 5 for the subspace $\mathbb{P}_2$. Find the best approximation to $p(t) = 5 - \frac{1}{2}t^4$ by polynomials in $\mathbb{P}_2$.

Solution: The values of p_0, p_1, and p_2 at the numbers $-2, -1, 0, 1$, and 2 are listed in $\mathbb{R}^5$ vectors in (4) above. The corresponding values for p are -3, $9/2$, 5, $9/2$, and -3. Compute

$$\langle p, p_0\rangle = (-3)\cdot 1 + \left(\frac{9}{2}\right)\cdot 1 + 5\cdot 1 + \left(\frac{9}{2}\right)\cdot 1 + (-3)\cdot 1 = 8$$

$$\langle p_0, p_0\rangle = 1\cdot 1 + 1\cdot 1 + 1\cdot 1 + 1\cdot 1 + 1\cdot 1 = 5$$

$$\langle p, p_1\rangle = (-3)\cdot(-2) + \left(\frac{9}{2}\right)\cdot(-1) + 5\cdot 0 + \left(\frac{9}{2}\right)\cdot 1 + (-3)\cdot 2 = 0$$

$$\langle p, p_2\rangle = (-3)\cdot(2) + \left(\frac{9}{2}\right)\cdot(-1) + 5\cdot(-2) + \left(\frac{9}{2}\right)\cdot(-1) + (-3)\cdot 2 = -31$$

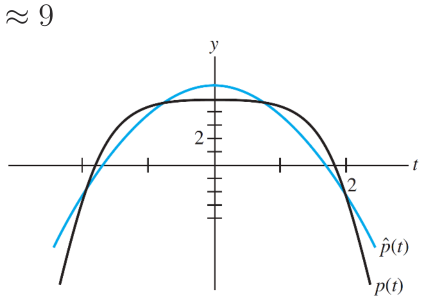
$$\langle p_2, p_2\rangle = 2\cdot 2 + (-1)\cdot(-1) + (-2)\cdot(-2) + (-1)\cdot(-1) + 2\cdot 2 = 14$$

Then the best approximation in V to p by polynomials in $\mathbb{P}_2$ is

$$\hat{p} = \operatorname{proj}_{\mathbb{P}_2} p = \frac{\langle p, p_0\rangle}{\langle p_0, p_0\rangle} p_0 + \frac{\langle p, p_1\rangle}{\langle p_1, p_1\rangle} p_1 + \frac{\langle p, p_2\rangle}{\langle p_2, p_2\rangle} p_2 = \frac{8}{5} p_0 + \frac{0}{\langle p_1, p_1\rangle} p_1 + \frac{-31}{14} p_2 = \frac{8}{5} - \frac{31}{14}(t^2 - 2)$$

This polynomial is the closest to p of all polynomials in $\mathbb{P}_2$, when the distance between polynomials is measured only at $-2, -1, 0, 1, 2$ and this distance is approximately 9:

$$\begin{aligned} \|p - \hat{p}\| = \sqrt{\langle p, \hat{p}\rangle} &= \sqrt{p(-2)\hat{p}(-2) + p(-1)\hat{p}(-1) + p(0)\hat{p}(0) + p(1)\hat{p}(-1) + p(2)\hat{p}(2)} \\ &= \sqrt{(-3)\left(-\frac{99}{35}\right) + \left(\frac{9}{2}\right)\left(\frac{267}{70}\right) + (5)\left(\frac{211}{35}\right) + \left(\frac{9}{2}\right)\left(\frac{267}{70}\right) + (-3)\left(-\frac{99}{35}\right)} \\ &= \frac{1}{50}\sqrt{399070} \approx 9 \end{aligned}$$



REMARK: The polynomials p_0, p_1, and p_2 in Examples 5 and 6 belong to a class of polynomials that are referred to in statistics as *orthogonal polynomials*.

Two Inequalities

THEOREM (The Cauchy-Schwarz Inequality): For all $\mathbf{u}, \mathbf{v}$ in V,

$$|\langle \mathbf{u}, \mathbf{v} \rangle| \leq \|\mathbf{u}\| \|\mathbf{v}\|$$

REMARK: In particular, setting

$$\mathbf{u} = (u_1, u_2), \quad \mathbf{v} = (v_1, v_2) \quad \text{and} \quad \langle \mathbf{u}, \mathbf{v} \rangle = u_1 v_1 + u_2 v_2$$

we get

$$|u_1 v_1 + u_2 v_2| \leq \sqrt{u_1^2 + u_2^2} \sqrt{v_1^2 + v_2^2}$$

which implies

$$(u_1 v_1 + u_2 v_2)^2 \leq (u_1^2 + u_2^2)(v_1^2 + v_2^2)$$

In general,

$$(u_1 v_1 + \ldots + u_n v_n)^2 \leq (u_1^2 + \ldots + u_n^2)(v_1^2 + \ldots + v_n^2)$$

THEOREM (The Triangle Inequality): For all $\mathbf{u}, \mathbf{v}$ in V,

$$\|\mathbf{u} + \mathbf{v}\| \leq \|\mathbf{u}\| + \|\mathbf{v}\|$$

An Inner Product for $C[a, b]$

EXAMPLE 7: For f, g in $C[a, b]$, set

$$\langle f, g \rangle = \int\limits_a^b f(t) g(t) dt \tag{5}$$

Show that (5) defines an inner product on $C[a, b]$.

Solution: Axiom 1 is satisfied, because

$$\langle f, g \rangle = \int\limits_a^b f(t) g(t) dt = \int\limits_a^b g(t) f(t) dt = \langle g, f \rangle$$

If h is in $C[a, b]$, then

$$\begin{aligned}
\langle f + g, h \rangle &= \int\limits_a^b (f(t) + g(t)) h(t) dt \\
&= \int\limits_a^b \Big(f(t) h(t) + g(t) h(t) \Big) dt \\
&= \int\limits_a^b f(t) h(t) dt + \int\limits_a^b g(t) h(t) dt \\
&= \langle f, h \rangle + \langle g, h \rangle
\end{aligned}$$

This verifies Axiom 2. For Axiom 3, compute

$$\langle cf, g\rangle = \int_a^b cf(t)g(t)dt = c\int_a^b f(t)g(t)dt = c\langle f, g\rangle$$

For Axiom 4, observe that

$$\langle f, f\rangle = \int_a^b [f(t)]^2 dt \geq 0$$

The function $[f(t)]^2$ is continuous and nonnegative on $[a, b]$. If the definite integral of $[f(t)]^2$ is zero, then $[f(t)]^2$ must be identically zero on $[a, b]$, by a theorem in advanced calculus, in which case f is the zero function. Thus $\langle f, f\rangle = 0$ implies that f is the zero function on $[a, b]$. So (5) defines an inner product on $C[a, b]$.

EXAMPLE 8: Let V be the space $C[0, 1]$ with the inner product of Example 7, and let W be the subspace spanned by the polynomials

$$p_1(t) = 1, \quad p_2(t) = 2t - 1, \quad \text{and} \quad p_3(t) = 12t^2$$

Use the Gram-Schmidt process to find an orthogonal basis for W.

Solution: Let $q_1 = p_1$, and compute

$$\langle p_2, q_1\rangle = \int_0^1 (2t - 1)(1)dt = (t^2 - t)\Big|_0^1 = 0$$

So p_2 is already orthogonal to q_1, and we can take $q_2 = p_2$. For the projection of p_3 onto $W_2 = \text{Span}\{q_1, q_2\}$, compute

$$\langle p_3, q_1\rangle = \int_0^1 12t^2(1)dt = 4t^3\Big|_0^1 = 4$$

$$\langle q_1, q_1\rangle = \int_0^1 1 \cdot 1dt = t\Big|_0^1 = 1$$

$$\langle p_3, q_2\rangle = \int_0^1 12t^2(2t - 1)dt = \int_0^1 (24t^3 - 12t^2)dt = (6t^4 - 4t^3)\Big|_0^1 = 2$$

$$\langle q_2, q_2\rangle = \int_0^1 (2t - 1)^2 dt = \frac{1}{6}(2t - 1)^3\Big|_0^1 = \frac{1}{3}$$

Then

$$\text{proj}_{W_2} p_3 = \frac{\langle p_3, q_1\rangle}{\langle q_1, q_1\rangle} q_1 + \frac{\langle p_3, q_2\rangle}{\langle q_2, q_2\rangle} q_2 = \frac{4}{1} q_1 + \frac{2}{1/3} q_2 = 4q_1 + 6q_2$$

and

$$q_3 = p_3 - \text{proj}_{W_2} p_3 = p_3 - 4q_1 - 6q_2$$

As a function,

$$q_3(t) = 12t^2 - 4 - 6(2t - 1) = 12t^2 - 12t + 2$$

for the subspace W is $\{q_1, q_2, q_3\}$.