PROBLEM:

Let

$$A = \begin{bmatrix} 4 & 0 \\ 0 & 2 \\ 1 & 1 \end{bmatrix}, \quad \bar{b} = \begin{bmatrix} 2 \\ 0 \\ 11 \end{bmatrix}.$$

Find a solution of $A\bar{x} = \bar{b}$.

DEFINITION:

Let A be an $m \times n$ matrix and $\bar{b}$ be in R^m. The general least-squares problem is the problem of finding an $\bar{x}$ that makes

$$\|\bar{b} - A\bar{x}\|$$

as small as possible. A least-squares solution of $A\bar{x} = \bar{b}$ is an $\hat{x}$ in R^n such that

$$\|\bar{b} - A\hat{x}\| \le \|\bar{b} - A\bar{x}\|$$

for all $\bar{x}$ in R^n.

THEOREM:

The set of least-squares solutions of $A\bar{x} = \bar{b}$ coincides with the nonempty set of solutions of the system

$$A^T A\bar{x} = A^T\bar{b}.$$

We usually call this system the normal equations.

EXAMPLE:

Let

$$A = \begin{bmatrix} 4 & 0 \\ 0 & 2 \\ 1 & 1 \end{bmatrix}, \quad \bar{b} = \begin{bmatrix} 2 \\ 0 \\ 11 \end{bmatrix}.$$

Find a least-squares solution of the inconsistent system $A\bar{x} = \bar{b}$.

SOLUTION:

We first compute $A^T A$ and $A^T\bar{b}$. We have

$$A^T A = \begin{bmatrix} 4 & 0 & 1 \\ 0 & 2 & 1 \end{bmatrix} \begin{bmatrix} 4 & 0 \\ 0 & 2 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 17 & 1 \\ 1 & 5 \end{bmatrix}$$

and

$$A^T\bar{b} = \begin{bmatrix} 4 & 0 & 1 \\ 0 & 2 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 0 \\ 11 \end{bmatrix} = \begin{bmatrix} 19 \\ 11 \end{bmatrix}.$$

Then the equation $A^T A\bar{x} = A^T\bar{b}$ becomes

$$\begin{bmatrix} 17 & 1 \\ 1 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 19 \\ 11 \end{bmatrix},$$

so

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 17 & 1 \\ 1 & 5 \end{bmatrix}^{-1} \begin{bmatrix} 19 \\ 11 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}.$$

PROBLEM:

Let

$$A = \begin{bmatrix} -1 & 2 \\ 2 & -3 \\ -1 & 3 \end{bmatrix}, \quad \bar{b} = \begin{bmatrix} 4 \\ 1 \\ 2 \end{bmatrix}.$$

Find a least-squares solution of the inconsistent system $A\bar{x} = \bar{b}$.

SOLUTION:

We first compute $A^T A$ and $A^T \bar{b}$. We have

$A^T A$

$$= \begin{bmatrix} -1 & 2 & -1 \\ 2 & -3 & 3 \end{bmatrix} \begin{bmatrix} -1 & 2 \\ 2 & -3 \\ -1 & 3 \end{bmatrix} = \begin{bmatrix} 6 & -11 \\ -11 & 22 \end{bmatrix}$$

and

$$A^T \bar{b} = \begin{bmatrix} -1 & 2 & -1 \\ 2 & -3 & 3 \end{bmatrix} \begin{bmatrix} 4 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} -4 \\ 11 \end{bmatrix}.$$

Then the equation $A^T A \bar{x} = A^T \bar{b}$ becomes

$$\begin{bmatrix} 6 & -11 \\ -11 & 22 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -4 \\ 11 \end{bmatrix},$$

so

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 6 & -11 \\ -11 & 22 \end{bmatrix}^{-1} \begin{bmatrix} -4 \\ 11 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}.$$