THEOREM(The Orthogonal Decomposition Theorem):

Let W be a subspace of R^n. Then each $\bar{y}$ in R^n can be written uniquely in the form

$$\bar{y} = \hat{y} + \bar{z},$$

where $\hat{y}$ is in W and $\bar{z}$ is in $W^{\perp}$. In fact, if $\{\bar{u}_1, \dots, \bar{u}_p\}$ is any orthogonal basis of W, then

$$\hat{y} = \frac{\bar{y} \cdot \bar{u}_1}{\bar{u}_1 \cdot \bar{u}_1}\bar{u}_1 + \dots + \frac{\bar{y} \cdot \bar{u}_p}{\bar{u}_p \cdot \bar{u}_p}\bar{u}_p$$

and $\bar{z} = \bar{y} - \hat{y}$.

DEFINITION:

The vector $\hat{y}$ is called the orthogonal projection of $\bar{y}$ onto W and written as

$$\text{proj}_W \bar{y}.$$

EXAMPLE:

Let

$$\bar{u}_1 = \begin{bmatrix} 2 \\ 5 \\ -1 \end{bmatrix}, \ \bar{u}_2 = \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}, \text{ and } \bar{y} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}.$$

(i) Find the orthogonal projection of $\bar{y}$ onto $W = \text{Span}\{\bar{u}_1, \bar{u}_2\}$;

(ii) Write $\bar{y}$ as the sum of a vector in W and a vector orthogonal to W.

SOLUTION:

(i) By the Theorem above, the orthogonal projection of $\bar{y}$ onto W is

$$\hat{y} = \frac{\bar{y} \cdot \bar{u}_1}{\bar{u}_1 \cdot \bar{u}_1}\bar{u}_1 + \frac{\bar{y} \cdot \bar{u}_2}{\bar{u}_2 \cdot \bar{u}_2}\bar{u}_2$$

$$= \frac{9}{30}\begin{bmatrix} 2 \\ 5 \\ -1 \end{bmatrix} + \frac{3}{6}\begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -2/5 \\ 2 \\ 1/5 \end{bmatrix}.$$

(ii) By the Theorem above we have $\bar{z} = \bar{y} - \hat{y}$, therefore

$$\bar{z} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} - \begin{bmatrix} -2/5 \\ 2 \\ 1/5 \end{bmatrix} = \begin{bmatrix} 7/5 \\ 0 \\ 14/5 \end{bmatrix}.$$

So,

$$\bar{y} = \begin{bmatrix} -2/5 \\ 2 \\ 1/5 \end{bmatrix} + \begin{bmatrix} 7/5 \\ 0 \\ 14/5 \end{bmatrix}.$$

PROBLEM:

Let

$$\bar{u}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \ \bar{u}_2 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \text{ and } \bar{y} = \begin{bmatrix} -1 \\ 4 \\ 3 \end{bmatrix}.$$

(i) Find the orthogonal projection of $\bar{y}$ onto $W = \text{Span}\{\bar{u}_1, \bar{u}_2\}$;

(ii) Write $\bar{y}$ as the sum of a vector in W and a vector orthogonal to W.

SOLUTION:

(i) By the Theorem above, the orthogonal projection of $\bar{y}$ onto W is

$$\hat{y} = \frac{\bar{y}\cdot\bar{u}_1}{\bar{u}_1\cdot\bar{u}_1}\bar{u}_1 + \frac{\bar{y}\cdot\bar{u}_2}{\bar{u}_2\cdot\bar{u}_2}\bar{u}_2$$

$$= \frac{3}{2}\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + \frac{5}{2}\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ 4 \\ 0 \end{bmatrix}.$$

(ii) By the Theorem above we have $\bar{z} = \bar{y} - \hat{y}$, therefore

$$\bar{z} = \begin{bmatrix} -1 \\ 4 \\ 3 \end{bmatrix} - \begin{bmatrix} -1 \\ 4 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 3 \end{bmatrix}.$$

So,

$$\bar{y} = \begin{bmatrix} -1 \\ 4 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 3 \end{bmatrix}.$$