DEFINITION:

A set of vectors $\{\bar{u}_1, \ldots, \bar{u}_p\}$ in R^n is said to be an orthogonal set if each pair of distinct vectors from the set is orthogonal, that is

$$\bar{u}_i \cdot \bar{u}_j = 0$$

for any $i \neq j$.

EXAMPLE:

Let

$$\bar{u}_1 = \begin{bmatrix} 3 \\ 0 \\ 0 \end{bmatrix}, \ \bar{u}_2 = \begin{bmatrix} 0 \\ 8 \\ 0 \end{bmatrix}, \ \bar{u}_3 = \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix}.$$

Then $\{\bar{u}_1, \ \bar{u}_2, \ \bar{u}_3\}$ is an orthogonal set.

PROBLEM:

Let

$$\bar{u}_1 = \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix}, \ \bar{u}_2 = \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}, \ \bar{u}_3 = \begin{bmatrix} -1/2 \\ -2 \\ 7/2 \end{bmatrix}.$$

Show that $\{\bar{u}_1, \ \bar{u}_2, \ \bar{u}_3\}$ is an orthogonal set.

SOLUTION:

We have

$$\bar{u}_1 \cdot \bar{u}_2 = 3(-1) + 1 \cdot 2 + 1 \cdot 1 = 0$$

$$\bar{u}_1 \cdot \bar{u}_3 = 3\left(-\frac{1}{2}\right) + 1(-2) + 1\left(\frac{7}{2}\right) = 0$$

$$\bar{u}_2 \cdot \bar{u}_3 = -1\left(-\frac{1}{2}\right) + 2(-2) + 1\left(\frac{7}{2}\right) = 0$$

THEOREM:

If $\mathcal{S} = \{\bar{u}_1, \ldots, \bar{u}_p\}$ is an orthogonal set of nonzero vectors in R^n, then $\mathcal{S}$ is linearly independent and hence is a basis (so-called, an orthogonal basis) for the subspace spanned by $\mathcal{S}$. Of course, if $p = n$, then $\mathcal{S}$ is a basis for R^n.

EXAMPLE:

Let $\mathcal{S} = \{\bar{u}_1, \ \bar{u}_2, \ \bar{u}_3\}$, where

$$\bar{u}_1 = \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix}, \ \bar{u}_2 = \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}, \ \bar{u}_3 = \begin{bmatrix} -1/2 \\ -2 \\ 7/2 \end{bmatrix}.$$

Then $\mathcal{S}$ is an orthogonal basis for R^3.

PROBLEM:

Let $\mathcal{S} = \{\bar{u}_1, \ \bar{u}_2, \ \bar{u}_3\}$, where

$$\bar{u}_1 = \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix}, \ \bar{u}_2 = \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}, \ \bar{u}_3 = \begin{bmatrix} -1/2 \\ -2 \\ 7/2 \end{bmatrix}.$$

Find coordinates of $\bar{y} = (6, 1, -8)$ in $\mathcal{S}$.

SOLUTION:

We have:

$$\begin{bmatrix} 3 & -1 & -1/2 & 6 \\ 1 & 2 & -2 & 1 \\ 1 & 1 & 7/2 & -8 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & -2 & 1 \\ 3 & -1 & -1/2 & 6 \\ 1 & 1 & 7/2 & -8 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 2 & -2 & 1 \\ 0 & -7 & 5.5 & 3 \\ 0 & -1 & 5.5 & -9 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & -2 & 1 \\ 0 & -1 & 5.5 & -9 \\ 0 & -7 & 5.5 & 3 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 2 & -2 & 1 \\ 0 & -1 & 5.5 & -9 \\ 0 & 0 & -33 & 66 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & -2 & 1 \\ 0 & 1 & -5.5 & 9 \\ 0 & 0 & 1 & -2 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 2 & 0 & -3 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & -2 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & -2 \end{bmatrix}$$

THEOREM:

Let $\mathcal{S} = \{\bar{u}_1, \ldots, \bar{u}_p\}$ be an orthogonal basis for a subspace W of R^n. For each $\bar{y}$ in W the weights in the linear combination

$$\bar{y} = c_1\bar{u}_1 + \ldots + c_p\bar{u}_p$$

are given by

$$c_j = \frac{\bar{y} \cdot \bar{u}_j}{\bar{u}_j \cdot \bar{u}_j} \quad (j = 1, \ldots, p).$$

PROOF:

Let $c_1, \ldots, c_p$ be such numbers that

$$\bar{y} = c_1\bar{u}_1 + c_2\bar{u}_2 + \ldots + c_p\bar{u}_p. \qquad (*)$$

If we multiply both sides of $(*)$ by $\bar{u}_1$, we get

$\bar{y} \cdot \bar{u}_1$

$= c_1\bar{u}_1 \cdot \bar{u}_1 + c_2\bar{u}_2 \cdot \bar{u}_1 + \ldots + c_p\bar{u}_p \cdot \bar{u}_1$

$= c_1\bar{u}_1 \cdot \bar{u}_1 + 0 + \ldots + 0$

$= c_1\bar{u}_1 \cdot \bar{u}_1$

because of orthogonality of $\bar{u}_1, \ldots, \bar{u}_p$. So, $\bar{y} \cdot \bar{u}_1 = c_1\bar{u}_1 \cdot \bar{u}_1$, therefore

$$c_1 = \frac{\bar{y} \cdot \bar{u}_1}{\bar{u}_1 \cdot \bar{u}_1}.$$

Similarly, if we multiply both sides of $(*)$ by $\bar{u}_j$, we deduce

$$c_j = \frac{\bar{y} \cdot \bar{u}_j}{\bar{u}_j \cdot \bar{u}_j} \quad (j = 1, \ldots, p).$$

PROBLEM:

Let $\mathcal{S} = \{\bar{u}_1,\ \bar{u}_2,\ \bar{u}_3\}$, where

$$\bar{u}_1 = \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix},\ \bar{u}_2 = \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix},\ \bar{u}_3 = \begin{bmatrix} -1/2 \\ -2 \\ 7/2 \end{bmatrix}.$$

Find coordinates of $\bar{y} = (6, 1, -8)$ in $\mathcal{S}$.

SOLUTION:

We have:

$\bar{y} \cdot \bar{u}_1 = 11, \ \bar{y} \cdot \bar{u}_2 = -12, \ \bar{y} \cdot \bar{u}_3 = -33$

and

$\bar{u}_1 \cdot \bar{u}_1 = 11, \ \bar{u}_2 \cdot \bar{u}_2 = 6, \ \bar{u}_3 \cdot \bar{u}_3 = 33/2,$

so

$$c_1 = \frac{\bar{y} \cdot \bar{u}_1}{\bar{u}_1 \cdot \bar{u}_1} = \frac{11}{11} = 1$$

$$c_2 = \frac{\bar{y} \cdot \bar{u}_2}{\bar{u}_2 \cdot \bar{u}_2} = \frac{-12}{6} = -2$$

$$c_3 = \frac{\bar{y} \cdot \bar{u}_3}{\bar{u}_3 \cdot \bar{u}_3} = \frac{-33}{33/2} = -2$$

therefore

$$[\bar{x}]_{\mathcal{S}} = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \\ -2 \end{bmatrix}.$$

AN ORTHOGONAL PROJECTION

PROBLEM:

Let $\bar{u}$ and $\bar{y}$ be nonzero vectors in R^n. Find vectors $\hat{y}$ and $\bar{z}$ such that

$$\bar{y} = \hat{y} + \bar{z},$$

where $\hat{y}$ is a multiple of $\bar{u}$ and $\bar{z}$ is orthogonal to $\bar{u}$.

SOLUTION:

Rewrite $\bar{y} = \hat{y} + \bar{z}$ as $\bar{z} = \bar{y} - \hat{y}$ and multiply both sides by $\bar{u}$:

$$\bar{z} \cdot \bar{u} = (\bar{y} - \hat{y}) \cdot \bar{u}$$

But $\bar{z}$ is orthogonal to $\bar{u}$, therefore

$$0 = (\bar{y} - \hat{y}) \cdot \bar{u}. \qquad (*)$$

Since $\hat{y}$ is a multiple of $\bar{u}$, we have

$$\hat{y} = \alpha \bar{u}, \text{ where } \alpha \text{ is a scalar.}$$

Substituting this into $(*)$, we get

$$0 = (\bar{y} - \alpha \bar{u}) \cdot \bar{u} = \bar{y} \cdot \bar{u} - \alpha \bar{u} \cdot \bar{u},$$

hence

$$\alpha = \frac{\bar{y} \cdot \bar{u}}{\bar{u} \cdot \bar{u}} \quad \text{and} \quad \hat{y} = \frac{\bar{y} \cdot \bar{u}}{\bar{u} \cdot \bar{u}} \bar{u}.$$

DEFINITION:

The vector $\hat{y}$ is called the orthogonal projection of $\bar{y}$ onto $\bar{u}$ and denoted by

$$\text{proj}_{\bar{u}} \bar{y}.$$

The vector $\bar{z}$ is called the component of $\bar{y}$ orthogonal to $\bar{u}$.

EXAMPLE:

Let $\bar{y} = \begin{bmatrix} 7 \\ 6 \end{bmatrix}$ and $\bar{u} = \begin{bmatrix} 4 \\ 2 \end{bmatrix}$. Find the orthogonal projection of $\bar{y}$ onto $\bar{u}$. Write $\bar{y}$ as a sum of two orthogonal vectors, one in Span $\{\bar{u}\}$ and one orthogonal to $\bar{u}$.

SOLUTION:

We first find the orthogonal projection of $\bar{y}$ onto $\bar{u}$. We have

$$\hat{y} = \frac{\bar{y}\cdot\bar{u}}{\bar{u}\cdot\bar{u}}\bar{u} = \frac{7\cdot 4+6\cdot 2}{4\cdot 4+2\cdot 2}\bar{u} = 2\begin{bmatrix}4\\2\end{bmatrix} = \begin{bmatrix}8\\4\end{bmatrix}.$$

We now find the component $\bar{z}$. We have

$$\bar{z} = \bar{y} - \hat{y} = \begin{bmatrix}7\\6\end{bmatrix} - \begin{bmatrix}8\\4\end{bmatrix} = \begin{bmatrix}-1\\2\end{bmatrix}.$$

Finally, we write $\bar{y}$ as a sum of two orthogonal vectors, one in Span $\{\bar{u}\}$ and one orthogonal to $\bar{u}$:

$$\begin{bmatrix}7\\6\end{bmatrix} = \begin{bmatrix}8\\4\end{bmatrix} + \begin{bmatrix}-1\\2\end{bmatrix}.$$

REMARK:

Note that the orthogonal projection of $\bar{y}$ onto $\bar{u}$ is exactly the same as the orthogonal projection of $\bar{y}$ onto $c\bar{u}$, where c is any nonzero scalar. Hence this projection is determined by the subspace L spanned by $\bar{u}$. Therefore sometimes we denote $\hat{y}$ by

$$\text{proj}_L\bar{y}.$$

So,

$$\hat{y} = \text{proj}_{\bar{u}}\bar{y} = \text{proj}_L\bar{y} = \frac{\bar{y}\cdot\bar{u}}{\bar{u}\cdot\bar{u}}\bar{u}.$$

DEFINITION:

A set of vectors $\{\bar{u}_1,\ldots,\bar{u}_p\}$ in R^n is said to be an orthonormal set if it is an orthogonal set of unit vectors.

A set of vectors $\{\bar{u}_1,\ldots,\bar{u}_p\}$ in R^n is said to be an orthonormal basis if it is an orthogonal basis of unit vectors.

EXAMPLE:

Let

$$\bar{e}_1 = \begin{bmatrix}1\\0\\0\end{bmatrix},\ \bar{e}_2 = \begin{bmatrix}0\\1\\0\end{bmatrix},\ \bar{e}_3 = \begin{bmatrix}0\\0\\1\end{bmatrix}.$$

Then $\{\bar{e}_1,\ \bar{e}_2,\ \bar{e}_3\}$ is the orthonormal basis for R^3.

EXAMPLE:

We know that

$$\bar{u}_1 = \begin{bmatrix}3\\1\\1\end{bmatrix},\ \bar{u}_2 = \begin{bmatrix}-1\\2\\1\end{bmatrix},\ \bar{u}_3 = \begin{bmatrix}-1/2\\-2\\7/2\end{bmatrix}.$$

is the orthogonal basis for R^3. Then

$$\bar{w}_1 = \frac{\bar{u}_1}{||\bar{u}_1||} = \frac{1}{\sqrt{11}}\begin{bmatrix}3\\1\\1\end{bmatrix}$$

$$\bar{w}_2 = \frac{\bar{u}_2}{||\bar{u}_2||} = \frac{1}{\sqrt{6}}\begin{bmatrix}-1\\2\\1\end{bmatrix}$$

$$\bar{w}_3 = \frac{\bar{u}_3}{||\bar{u}_3||} = \sqrt{\frac{2}{33}}\begin{bmatrix}-1/2\\-2\\7/2\end{bmatrix} = \frac{1}{\sqrt{66}}\begin{bmatrix}-1\\-4\\7\end{bmatrix}$$

is the orthonormal basis for R^3.