THE INNER PRODUCT

DEFINITION:

If $\bar{u}$ and $\bar{v}$ are vectors in R^n, then $\bar{u}^T\bar{v}$ is called the inner product (or dot product) of $\bar{u}$ and $\bar{v}$ and written as

$$\bar{u}\cdot\bar{v}$$

REMARK:

In other words, if

$$\bar{u}=\begin{bmatrix}u_1\\ \vdots\\ u_n\end{bmatrix}\quad\text{and}\quad\bar{v}=\begin{bmatrix}v_1\\ \vdots\\ v_n\end{bmatrix},$$

then

$$\bar{u}\cdot\bar{v}=\bar{u}^T\bar{v}=[u_1\ \ldots\ u_n]\begin{bmatrix}v_1\\ \vdots\\ v_n\end{bmatrix}$$
$$=u_1v_1+\ldots+u_nv_n.$$

EXAMPLE:

Let

$$\bar{u}=\begin{bmatrix}2\\ -5\\ -1\end{bmatrix}\quad\text{and}\quad\bar{v}=\begin{bmatrix}3\\ 2\\ -3\end{bmatrix}.$$

Find $\bar{u}\cdot\bar{v}$.

SOLUTION:

We have

$\bar{u}\cdot\bar{v}=2\cdot3+(-5)\cdot2+(-1)(-3)=-1.$

THEOREM:

Let $\bar{u}$, $\bar{v}$, and $\bar{w}$ be vectors in R^n, and let c be a scalar. Then

(a) $\bar{u}\cdot\bar{v}=\bar{v}\cdot\bar{u}$

(b) $(\bar{u}+\bar{v})\cdot\bar{w}=\bar{u}\cdot\bar{w}+\bar{v}\cdot\bar{w}$

(c) $(c\bar{u})\cdot\bar{v}=c(\bar{u}\cdot\bar{v})=\bar{u}\cdot(c\bar{v})$

(d) $\bar{u}\cdot\bar{u}\geq 0$

(d′) $\bar{u}\cdot\bar{u}=0$ if and only if $\bar{u}=0$

THE LENGTH OF A VECTOR

DEFINITION:

Let $\bar{v}=(v_1,\ldots,v_n)$ be a vector from R^n. Then the length (or norm) of $\bar{v}$ is the nonnegative scalar $\|\bar{v}\|$ defined by

$$\|\bar{v}\|=\sqrt{\bar{v}\cdot\bar{v}}=\sqrt{v_1^2+\ldots+v_n^2}.$$

EXAMPLE:

The length of the vector

$$\bar{u}=\begin{bmatrix}3\\ 4\end{bmatrix}$$

is

$$\|\bar{u}\|=\sqrt{3^2+4^2}=\sqrt{25}=5.$$

PROPERTY:

Let c be a scalar. Then

$$\|c\bar{v}\| = |c|\|\bar{v}\|.$$

PROOF 1:

We have

$$\|c\bar{v}\| = \sqrt{(cv_1)^2 + \ldots + (cv_n)^2}$$

$$= \sqrt{c^2(v_1^2 + \ldots + v_n^2)}$$

$$= |c|\sqrt{v_1^2 + \ldots + v_n^2}$$

$$= |c|\|\bar{v}\|.$$

PROOF 2:

We have

$$\|c\bar{v}\|^2 = (c\bar{v}) \cdot (c\bar{v}) = c^2\bar{v} \cdot \bar{v} = c^2\|\bar{v}\|^2.$$

DEFINITION:

A vector whose length is 1 is called a unit vector.

PROBLEM:

Let $\bar{v} = (1, -2, 2, 0)$. Find:

(a) The length of $\bar{v}$;

(b) The unit vector in the same direction as $\bar{v}$.

SOLUTION:

(a) We have

$$\|\bar{v}\| = \sqrt{1^2 + (-2)^2 + 2^2 + 0^2} = \sqrt{9} = 3.$$

(b) Put

$$\bar{u} = \frac{1}{\|\bar{v}\|}\bar{v}.$$

Note that vectors $\bar{v}$ and $\bar{u}$ have the same direction. Moreover, since

$$\|\bar{u}\| = \left\| \frac{1}{\|\bar{v}\|}\bar{v} \right\| = \frac{1}{\|\bar{v}\|}\|\bar{v}\| = 1,$$

it follows that $\bar{u}$ is the unit vector. Finally, we have

$$\|\bar{u}\| = \frac{1}{\|\bar{v}\|}\bar{v} = \frac{1}{3}\begin{bmatrix} 1 \\ -2 \\ 2 \\ 0 \end{bmatrix} = \begin{bmatrix} 1/3 \\ -2/3 \\ 2/3 \\ 0 \end{bmatrix}.$$

DISTANCE IN R^n

DEFINITION:

Let $\bar{u}$ and $\bar{v}$ be from R^n. Then the distance between $\bar{u}$ and $\bar{v}$, written as

$$\text{dist } (\bar{u}, \bar{v}),$$

is the length of the vector $\bar{u} - \bar{v}$. That is,

$$\text{dist } (\bar{u}, \bar{v}) = \|\bar{u} - \bar{v}\|.$$

EXAMPLE:

Let

$$\bar{u} = (1, 2, 3) \quad \text{and} \quad \bar{v} = (-1, 5, -4).$$

Find the distance between $\bar{u}$ and $\bar{v}$.

SOLUTION:

Step 1: We have

$$\bar{u}-\bar{v}=(1,2,3)-(-1,5,-4)=(2,-3,7).$$

Step 2: By the Definiton above, we get

$$\text{dist }(\bar{u},\bar{v})=\sqrt{2^2+(-3)^2+7^2}=\sqrt{62}.$$

ORTHOGONAL VECTORS

DEFINITION:

Two vectors $\bar{u}$ and $\bar{v}$ in R^n are orthogonal (perpendicular) if

$$\bar{u}\cdot\bar{v}=0.$$

EXAMPLE:

Vectors $\bar{u}=(4,12)$ and $\bar{v}=(9,-3)$ are orthogonal, since

$$\bar{u}\cdot\bar{v}=4\cdot 9+12\cdot(-3)=0.$$

THEOREM (The Pythagorean Theorem):

Two vectors $\bar{u}$ and $\bar{v}$ in R^n are orthogonal if and only if

$$\|\bar{u}+\bar{v}\|^2=\|\bar{u}\|^2+\|\bar{v}\|^2.$$

ANGLES IN R^2 AND R^3

THEOREM:

Let $\bar{u}$ and $\bar{v}$ be from R^2 or R^3 and let θ be the angle between them. Then

$$\cos\theta=\frac{\bar{u}\cdot\bar{v}}{\|\bar{u}\|\|\bar{v}\|}$$

EXAMPLE:

Find the angle between vectors

$$\bar{u}=\begin{bmatrix}5\\-3\\1\end{bmatrix}\quad\text{and}\quad\bar{v}=\begin{bmatrix}6\\9\\-3\end{bmatrix}.$$

SOLUTION:

We have $\cos\theta=\dfrac{\bar{u}\cdot\bar{v}}{\|\bar{u}\|\|\bar{v}\|}=$

$$\frac{5\cdot 6+(-3)\cdot 9+1\cdot(-3)}{\sqrt{5^2+(-3)^2+1^2}\sqrt{6^2+9^2+(-3)^2}}=0,$$

therefore

$$\theta=\frac{\pi}{2}=90^\circ.$$

ORTHOGONAL COMPLEMENTS

DEFINITION:

If a vector $\bar{z}$ is orthogonal to every vector in a subspace W of R^n, then $\bar{z}$ is said to be orthogonal to W.

EXAMPLE:

Let $\bar{z} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$ and

$$W = \left\{ \begin{bmatrix} 6 \\ -4 \end{bmatrix} t : \; t \in R \right\}$$

be a subspace of R^2. Then $\bar{z}$ is orthogonal to every vector in W, since

$$\bar{z} \cdot \left(\begin{bmatrix} 6 \\ -4 \end{bmatrix} t \right) = \left(\begin{bmatrix} 2 \\ 3 \end{bmatrix} \cdot \begin{bmatrix} 6 \\ -4 \end{bmatrix} \right) t = 0\, t = 0.$$

EXAMPLE:

Let $\bar{z} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ and

$$W = \left\{ \begin{bmatrix} -1 \\ 5 \\ -3 \end{bmatrix} t_1 + \begin{bmatrix} 7 \\ 4 \\ -5 \end{bmatrix} t_2 : \; t_1, t_2 \in R \right\}$$

be a subspace of R^3. Then $\bar{z}$ is orthogonal to every vector in W, since

$$\bar{z} \cdot \left(\begin{bmatrix} -1 \\ 5 \\ -3 \end{bmatrix} t_1 + \begin{bmatrix} 7 \\ 4 \\ -5 \end{bmatrix} t_2 \right)$$

$$= \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \cdot \begin{bmatrix} -1 \\ 5 \\ -3 \end{bmatrix} t_1 + \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \cdot \begin{bmatrix} 7 \\ 4 \\ -5 \end{bmatrix} t_2$$

$$= 0\; t_1 + 0\; t_2 = 0.$$

DEFINITION:

The set of all vectors $\bar{z}$ that are orthogonal to W is called the orthogonal complement of W and is denoted by $W^{\perp}$.

EXAMPLE:

Let

$$H = \left\{ \begin{bmatrix} 2 \\ 3 \end{bmatrix} s : \; s \in R \right\}$$

and

$$W = \left\{ \begin{bmatrix} 6 \\ -4 \end{bmatrix} t : \; t \in R \right\}$$

be subspaces of R^2. Then every vector in H is orthogonal to every vector in W, since

$$\left(\begin{bmatrix} 2 \\ 3 \end{bmatrix} s \right) \cdot \left(\begin{bmatrix} 6 \\ -4 \end{bmatrix} t \right) = \left(\begin{bmatrix} 2 \\ 3 \end{bmatrix} \cdot \begin{bmatrix} 6 \\ -4 \end{bmatrix} \right) st = 0.$$

Moreover, one can show that there are no other vectors in R^2 which are orthogonal to every vector in W. Therefore $H = W^{\perp}$.

EXAMPLE:

Let L_1 be a line through the origin in R^2, and let L_2 be the line through the origin and perpendicular to L_1. Then each vector on L_1 is orthogonal to every vector in L_2. Moreover, one can show that there are no other vectors in R^2 which are orthogonal to every vector in L_1. Therefore

$$L_1 = L_2^{\perp}.$$

Also, for the same reason we have

$$L_2 = L_1^{\perp}.$$

EXAMPLE:

Let $H = \left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} s : \ s \in R \right\}$ and

$$W = \left\{ \begin{bmatrix} -1 \\ 5 \\ -3 \end{bmatrix} t_1 + \begin{bmatrix} 7 \\ 4 \\ -5 \end{bmatrix} t_2 : \ t_1, t_2 \in R \right\}$$

be subspaces of R^3. Then every vector in H is orthogonal to every vector in W, since

$$\left(\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} s \right) \cdot \left(\begin{bmatrix} -1 \\ 5 \\ -3 \end{bmatrix} t_1 + \begin{bmatrix} 7 \\ 4 \\ -5 \end{bmatrix} t_2 \right)$$

$$= \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \cdot \begin{bmatrix} -1 \\ 5 \\ -3 \end{bmatrix} s \ t_1 + \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \cdot \begin{bmatrix} 7 \\ 4 \\ -5 \end{bmatrix} s \ t_2$$

$$= 0 \ s \ t_1 + 0 \ s \ t_2 = 0.$$

Moreover, one can show that there are no other vectors in R^3 which are orthogonal to every vector in W. Therefore $H = W^{\perp}$.

EXAMPLE:

Let W be a plane through the origin in R^3, and let L be the line through the origin and perpendicular to W. Then each vector on L is orthogonal to every vector $\bar{z}$ in W. Moreover, one can show that there are no other vectors in R^3 which are orthogonal to every vector in W. Therefore

$$L = W^{\perp}.$$

Also, for the same reason we have

$$W = L^{\perp}.$$

THEOREM:

(a) A vector $\bar{x}$ is in $W^{\perp}$ if and only if $\bar{x}$ is orthogonal to every vector in a set that spans W.

(b) $W^{\perp}$ is a subspace of R^n.

THEOREM:

Let A be an $m \times n$ matrix. Then

$$(\text{Row } A)^{\perp} = \text{Nul } A$$

and

$$(\text{Col } A)^{\perp} = \text{Nul } A^T.$$

PROOF:

The row-column rule for computing $A\bar{x}$ shows that if $\bar{x}$ is in Nul A, then $\bar{x}$ is orthogonal to each row of A. Since the rows of A span the row space, $\bar{x}$ is orthogonal to Row A.

Conversely, if $\bar{x}$ is orthogonal to Row A, then $\bar{x}$ is certainly orthogonal to each row of A, and therefore we have $A\bar{x} = \bar{0}$.

To prove the second part of the theorem, we note that

$$(\text{Row } A^T)^{\perp} = \text{Nul } A^T \qquad (*)$$

by the first part of this theorem. On the other hand, it is easy to see that

$$\text{Row } A^T = \text{Col } A,$$

therefore

$$(\text{Row } A^T)^{\perp} = (\text{Col } A)^{\perp}. \qquad (**)$$

Combination of $(*)$ and $(**)$ gives the desired result. ■