PROBLEM:

Let $\mathcal{E} = \{\bar{e}_1,\ \bar{e}_2\}$ be a basis for a vector space $V$. Let also

$$T : V \to V$$

be a linear transformation such that

$$T(\bar{e}_1) = 3\bar{e}_1 - 2\bar{e}_2,$$

$$T(\bar{e}_2) = 4\bar{e}_1 + 7\bar{e}_2.$$

Find the matrix $M$ for the linear transformation $T$ relative to the basis $\mathcal{E}$.

SOLUTION:

Suppose

$$\bar{x} = x_1\bar{e}_1 + x_2\bar{e}_2$$

then

$$\begin{aligned} T(\bar{x}) &= T(x_1\bar{e}_1 + x_2\bar{e}_2) \\ &= T(x_1\bar{e}_1) + T(x_2\bar{e}_2) \\ &= x_1T(\bar{e}_1) + x_2T(\bar{e}_2) \\ &= x_1(3\bar{e}_1 - 2\bar{e}_2) + x_2(4\bar{e}_1 + 7\bar{e}_2) \\ &= (3x_1+4x_2)\bar{e}_1+(-2x_1+7x_2)\bar{e}_2 \end{aligned}$$

Therefore

$$T(\bar{x}) = \begin{bmatrix} 3x_1 + 4x_2 \\ -2x_1 + 7x_2 \end{bmatrix} = \begin{bmatrix} 3 & 4 \\ -2 & 7 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

So,

$$M = \begin{bmatrix} 3 & 4 \\ -2 & 7 \end{bmatrix}.$$

PROBLEM:

Let $\mathcal{B} = \{\bar{b}_1,\ \bar{b}_2\}$ be a basis for $V$ and $\mathcal{C} = \{\bar{c}_1,\ \bar{c}_2,\ \bar{c}_3\}$ be a basis for $W$. Let also

$$T : V \to W$$

be a linear transformation such that

$$T(\bar{b}_1) = 3\bar{c}_1 - 2\bar{c}_2 + 5\bar{c}_3,$$

$$T(\bar{b}_2) = 4\bar{c}_1 + 7\bar{c}_2 - \bar{c}_3.$$

Find the matrix $M$ for the linear transformation $T$ relative to the bases $\mathcal{B}$ and $\mathcal{C}$.

SOLUTION:

Suppose

$$\bar{x} = x_1\bar{b}_1 + x_2\bar{b}_2$$

then

$$\begin{aligned} &T(\bar{x}) \\ &= T(x_1\bar{b}_1 + x_2\bar{b}_2) \\ &= T(x_1\bar{b}_1) + T(x_2\bar{b}_2) \\ &= x_1T(\bar{b}_1) + x_2T(\bar{b}_2) \\ &= x_1(3\bar{c}_1 - 2\bar{c}_2 + 5\bar{c}_3) + x_2(4\bar{c}_1 + 7\bar{c}_2 - \bar{c}_3) \\ &= (3x_1+4x_2)\bar{c}_1+(-2x_1+7x_2)\bar{c}_2+(5x_1-x_2)\bar{c}_3 \end{aligned}$$

So, $T(\bar{x}) =$

$(3x_1+4x_2)\bar{c}_1+(-2x_1+7x_2)\bar{c}_2+(5x_1-x_2)\bar{c}_3$

Hence

$$T(\bar{x}) = \begin{bmatrix} 3x_1 + 4x_2 \\ -2x_1 + 7x_2 \\ 5x_1 - x_2 \end{bmatrix}$$

$$= \begin{bmatrix} 3 & 4 \\ -2 & 7 \\ 5 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Therefore

$$M = \begin{bmatrix} 3 & 4 \\ -2 & 7 \\ 5 & -1 \end{bmatrix}.$$

<u>THEOREM</u>:

Let $V$ be an $n$-dimensional vector space and $W$ be an $m$-dimensional vector space and let

$$T : V \to W$$

be a linear transformation. Let also $\mathcal{B} = \{\bar{b}_1, \ldots, \bar{b}_n\}$ and $\mathcal{C} = \{\bar{c}_1, \ldots, \bar{c}_m\}$ be bases for $V$ and $W$ respectively. Then

$$[T(\bar{x})]_{\mathcal{C}} = M\,[\bar{x}]_{\mathcal{B}},$$

where $[\bar{x}]_{\mathcal{B}}$ is the coordinate vector of $\bar{x}$ in the basis $\mathcal{B}$, $[T(\bar{x})]_{\mathcal{C}}$ is the coordinate vector of its image in the basis $\mathcal{C}$, and

$$M = \Big[ [T(\bar{b}_1)]_{\mathcal{C}} \ \cdots \ [T(\bar{b}_n)]_{\mathcal{C}} \Big].$$

The matrix $M$ is called the <u>matrix for $T$ relative to the bases $\mathcal{B}$ and $\mathcal{C}$.</u>

<u>EXAMPLE</u>:

Let $\mathcal{B} = \{\bar{b}_1,\ \bar{b}_2,\ \bar{b}_3\}$ be a basis for $V$ and $\mathcal{C} = \{\bar{c}_1,\ \bar{c}_2,\ \bar{c}_3,\ \bar{c}_4\}$ be a basis for $W$. Let also

$$T : V \to W$$

be a linear transformation such that

$$T(\bar{b}_1) = \bar{c}_1 + 5\bar{c}_2,$$

$$T(\bar{b}_2) = 3\bar{c}_3 + 4\bar{c}_4,$$

$$T(\bar{b}_3) = \bar{c}_2 - 5\bar{c}_3.$$

Find the matrix $M$ for the linear transformation $T$ relative to the bases $\mathcal{B}$ and $\mathcal{C}$.

<u>SOLUTION</u>:

The coordinates of $T(\bar{b}_1)$, $T(\bar{b}_2)$, and $T(\bar{b}_3)$ in $\mathcal{C}$ are

$$[T(\bar{b}_1)]_{\mathcal{C}} = \begin{bmatrix} 1 \\ 5 \\ 0 \\ 0 \end{bmatrix} \qquad [T(\bar{b}_2)]_{\mathcal{C}} = \begin{bmatrix} 0 \\ 0 \\ 3 \\ 4 \end{bmatrix}$$

and

$$[T(\bar{b}_3)]_{\mathcal{C}} = \begin{bmatrix} 0 \\ 1 \\ -5 \\ 0 \end{bmatrix}$$

therefore by the Theorem above we have

$$M = \begin{bmatrix} 1 & 0 & 0 \\ 5 & 0 & 1 \\ 0 & 3 & -5 \\ 0 & 4 & 0 \end{bmatrix}.$$

**EXAMPLE**:

Let
$$T : P_2 \to P_2$$
be a linear transformation such that
$$T(a_0 + a_1 t + a_2 t^2) = a_1 + 2a_2 t.$$

(a) Find the matrix $M$ of the linear transformation $T$ for the standard basis
$$\mathcal{E} = \{1,\ t,\ t^2\}.$$

(b)Verify that
$$[T(\bar{p})]_{\mathcal{E}} = M[\bar{p}]_{\mathcal{E}}$$
for any $\bar{p} \in P_2$.

(c) Find the matrix of the linear transformation $T$ relative to
$$\mathcal{B} = \{1,\ t,\ t - t^2\}.$$
and $\mathcal{E}$.

**SOLUTION**:

(a) We have
$$T(1) = 0,\ T(t) = 1,\ T(t^2) = 2t,$$
therefore
$$[T(1)]_{\mathcal{E}} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \qquad [T(t)]_{\mathcal{E}} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$
and
$$[T(t^2)]_{\mathcal{E}} = \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix},$$
therefore by the Theorem above we have
$$M = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix}.$$

(b) Let
$$\bar{p} = a_0 + a_1 t + a_2 t^2.$$
We have
$$\begin{aligned}[T(\bar{p})]_{\mathcal{E}} = [a_1 + 2a_2 t]_{\mathcal{E}} &= \begin{bmatrix} a_1 \\ 2a_2 \\ 0 \end{bmatrix} \\ &= \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix} \\ &= M[\bar{p}]_{\mathcal{E}}\end{aligned}$$

(c) We have
$$T(1) = 0,\ T(t) = 1,\ T(t - t^2) = 1 - 2t,$$
therefore
$$[T(1)]_{\mathcal{E}} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \qquad [T(t)]_{\mathcal{E}} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$
and
$$[T(t - t^2)]_{\mathcal{E}} = \begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix},$$
therefore by the Theorem above we have
$$M = \begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & -2 \\ 0 & 0 & 0 \end{bmatrix}.$$