PROBLEM:

Let

$$A = \begin{bmatrix} 7 & 4 \\ -3 & -1 \end{bmatrix}, \quad D = \begin{bmatrix} 1 & 0 \\ 0 & 5 \end{bmatrix}.$$

Find a formula for $A^k$ and $D^k$.

SOLUTION:

(a) We have

$$D^2 = \begin{bmatrix} 1 & 0 \\ 0 & 5 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 5 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 5^2 \end{bmatrix}$$

$$D^3 = D^2 D = \begin{bmatrix} 1 & 0 \\ 0 & 5^2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 5 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 5^3 \end{bmatrix}$$

In general,

$$D^k = \begin{bmatrix} 1^k & 0 \\ 0 & 5^k \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 5^k \end{bmatrix}$$

(b) We have

$$A^2 = \begin{bmatrix} 7 & 4 \\ -3 & -1 \end{bmatrix} \begin{bmatrix} 7 & 4 \\ -3 & -1 \end{bmatrix} = \begin{bmatrix} 37 & 24 \\ -18 & -11 \end{bmatrix}$$

$$A^3 = \begin{bmatrix} 37 & 24 \\ -18 & -11 \end{bmatrix} \begin{bmatrix} 7 & 4 \\ -3 & -1 \end{bmatrix} = \begin{bmatrix} 187 & 124 \\ -93 & -61 \end{bmatrix}$$

$$A^4 = \begin{bmatrix} 187 & 124 \\ -93 & -61 \end{bmatrix} \begin{bmatrix} 7 & 4 \\ -3 & -1 \end{bmatrix} = \begin{bmatrix} 937 & 624 \\ -468 & -311 \end{bmatrix}$$

$$A^5 = \begin{bmatrix} 937 & 624 \\ -468 & -311 \end{bmatrix} \begin{bmatrix} 7 & 4 \\ -3 & -1 \end{bmatrix} = \begin{bmatrix} 4687 & 3124 \\ -2343 & -1561 \end{bmatrix}$$

Let

$$P = \begin{bmatrix} -2 & -2 \\ 3 & 1 \end{bmatrix}$$

One can check that

$$A = PDP^{-1}.$$

We have

$$\begin{aligned} A^2 &= PDP^{-1}PDP^{-1} \\ &= PD\underbrace{(P^{-1}P)}_{I}DP^{-1} \\ &= PDDP^{-1} = PD^2P^{-1} \end{aligned}$$

Similarly,

$$A^3 = PD^3P^{-1}$$
$$A^4 = PD^4P^{-1}$$
$$A^5 = PD^5P^{-1}$$

In general,

$$A^k = PD^kP^{-1} = \begin{bmatrix} -2 & -2 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 5^k \end{bmatrix} \begin{bmatrix} 1/4 & 1/2 \\ -3/4 & -1/2 \end{bmatrix}$$

DEFINITION:

A square matrix $A$ is said to be diagonalizable if $A$ is similar to a diagonal matrix, that is

$$A = PDP^{-1}$$

for some invertible matrix $P$ and some diagonal matrix $D$.

EXAMPLE:

Matrices

$$A = \begin{bmatrix} 7 & 4 \\ -3 & -1 \end{bmatrix} \text{ and } D = \begin{bmatrix} 1 & 0 \\ 0 & 5 \end{bmatrix}$$

are similar, since

$$A = PDP^{-1},$$

where

$$P = \begin{bmatrix} -2 & -2 \\ 3 & 1 \end{bmatrix}.$$

Also, $A$ is diagonalizable.

**THEOREM** (The Diagonalization Theorem):
An $n \times n$ matrix $A$ is diagonalizable if and only if $A$ has $n$ linearly independent eigenvectors. In this case:

(a) The columns of $P$ are $n$ linearly independent eigenvectors of $A$;

(b) The diagonal entries of $D$ are eigenvalues of $A$ that correspond, respectively, to the eigenvectors in $P$.

**EXAMPLE**:
One can check that $\lambda = 1, 5$ are eigenvalues of $A$ and

$$\begin{bmatrix} -2 \\ 3 \end{bmatrix} \quad \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

are corresponding eigenvectors. Therefore

$$D = \begin{bmatrix} 1 & 0 \\ 0 & 5 \end{bmatrix} \text{ and } P = \begin{bmatrix} -2 & -2 \\ 3 & 1 \end{bmatrix}.$$

**EXAMPLE**:

Determine if the following matrix is diagonalizable:

$$A = \begin{bmatrix} 1 & 3 & 3 \\ -3 & -5 & -3 \\ 3 & 3 & 1 \end{bmatrix}$$

**SOLUTION**:
We first solve the following equation:

$$\det(A - \lambda I) = \begin{vmatrix} 1-\lambda & 3 & 3 \\ -3 & -5-\lambda & -3 \\ 3 & 3 & 1-\lambda \end{vmatrix} = 0.$$

Expanding this determinant, we obtain

$$-\lambda^3 - 3\lambda^2 + 4 = (1-\lambda)(\lambda+2)^2 = 0,$$

hence

$$\lambda_1 = 1, \quad \lambda_2 = -2$$

are eigenvalues of $A$, so

$$D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{bmatrix}$$

One can show that

$$\text{Basis for } \lambda_1 = 1 : \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

$$\text{Basis for } \lambda_2 = -2 : \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \quad \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

therefore

$$P = \begin{bmatrix} 1 & -1 & -1 \\ -1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$