DEFINITION:
An eigenvector of an $n \times n$ matrix A is a nonzero vector $\bar{x}$ such that

$$A\bar{x} = \lambda\bar{x} \qquad (*)$$

for some scalar λ. A scalar λ is called an eigenvalue of A.

DEFINITION:
Let λ be an eigenvalue of A. The set of all solutions of $(*)$ is called the eigenspace of A corresponding to λ.

REMARK:
To find eigenvalues of A, we should solve the following characteristic equation

$$\det(A - \lambda I) = 0,$$

where I is the identity matrix.

Let

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{bmatrix}$$

then $\det(A - \lambda I) =$

$$\begin{vmatrix} a_{11} - \lambda & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} - \lambda & \dots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} - \lambda \end{vmatrix}$$

is called the characteristic polynomial of A and

$$\det(A - \lambda I) = 0,$$

is called the characteristic equation of A.

PROBLEM:
Let

$$A = \begin{bmatrix} 5 & 0 \\ 2 & 1 \end{bmatrix}.$$

Find all eigenvalues, eigenvectors and bases for the corresponding eigenspaces.

SOLUTION:
We first solve the following equation:

$$\det(A - \lambda I) = \begin{vmatrix} 5 - \lambda & 0 \\ 2 & 1 - \lambda \end{vmatrix} = 0.$$

Expanding this determinant, we obtain

$$(5 - \lambda)(1 - \lambda) = 0,$$

hence

$$\lambda_1 = 1, \quad \lambda_2 = 5$$

are eigenvalues of A.

(a) Let $\lambda = 1$. To solve the homogeneous system

$$(A - \lambda I)\bar{x} = \bar{0},$$

we use row operations:

$$\begin{bmatrix} 5 - \lambda & 0 & 0 \\ 2 & 1 - \lambda & 0 \end{bmatrix} = \begin{bmatrix} 4 & 0 & 0 \\ 2 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

hence

$$x_1 = 0.$$

We get

$$\bar{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ x_2 \end{bmatrix} = x_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

is the eigenvector of A, corresponding to $\lambda = 1$.

The 1-dimensional eigenspace corresponding to $\lambda = 1$ is

$$H = \left\{ t \begin{bmatrix} 0 \\ 1 \end{bmatrix} : t \text{ is any real number} \right\}$$

and $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ is the basis for H.

(b) Let $\lambda = 5$. To solve the homogeneous system

$$(A - \lambda I)\bar{x} = \bar{0},$$

we use row operations:

$$\begin{bmatrix} 5-\lambda & 0 & 0 \\ 2 & 1-\lambda & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 2 & -4 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

hence

$$x_1 - 2x_2 = 0 \quad \Longrightarrow \quad x_1 = 2x_2.$$

We get

$$\bar{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2x_2 \\ x_2 \end{bmatrix} = x_2 \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

is the eigenvector of A, corresponding to $\lambda = 5$.

The 1-dimensional eigenspace corresponding to $\lambda = 5$ is

$$H = \left\{ t \begin{bmatrix} 2 \\ 1 \end{bmatrix} : t \text{ is any real number} \right\}$$

and $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$ is the basis for H.

PROBLEM:

Let

$$A = \begin{bmatrix} 4 & -1 & 6 \\ 2 & 1 & 6 \\ 2 & -1 & 8 \end{bmatrix}.$$

An eigenvalue λ is 2. Find a basis for the corresponding eigenspace.

SOLUTION:

We use row operations:

$$\begin{bmatrix} 4-\lambda & -1 & 6 & 0 \\ 2 & 1-\lambda & 6 & 0 \\ 2 & -1 & 8-\lambda & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & -1 & 6 & 0 \\ 2 & -1 & 6 & 0 \\ 2 & -1 & 6 & 0 \end{bmatrix} \sim \begin{bmatrix} 2 & -1 & 6 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

hence

$$2x_1 - x_2 + 6x_3 = 0 \quad \Longrightarrow \quad x_1 = \frac{1}{2}x_2 - 3x_3$$

We get

$$\bar{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \frac{1}{2}x_2 - 3x_3 \\ x_2 \\ x_3 \end{bmatrix}$$

is the eigenvector of A, corresponding to $\lambda = 2$.

To find a basis for the eigenspace corresponding to $\lambda = 2$, we note that

$$\bar{x} = \begin{bmatrix} \frac{1}{2}x_2 - 3x_3 \\ x_2 \\ x_3 \end{bmatrix} = x_2 \begin{bmatrix} 1/2 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix}$$

therefore the 2-dimensional eigenspace corresponding to $\lambda = 2$ is

$$H = \left\{ t_1 \begin{bmatrix} 1/2 \\ 1 \\ 0 \end{bmatrix} + t_2 \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix} : t_1,\ t_2 \in R \right\}$$

and

$$\left\{ \begin{bmatrix} 1/2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix} \right\}$$

is the basis for H.

DEFINITION:
If A and B are $n \times n$ matrices, then A is similar to B if there is an invertible matrix P such that

$$P^{-1}AP = B$$

or, equivalently,

$$A = PBP^{-1}$$

THEOREM:
If $n \times n$ matrices A and B are similar, then they have the same characteristic polynomial and hence the same eigenvalues (with the same multiplicities).

Proof: If $B = P^{-1}AP$, then

$$\begin{aligned} B - \lambda I &= P^{-1}AP - \lambda P^{-1}P \\ &= P^{-1}(AP - \lambda P) \\ &= P^{-1}(A - \lambda I)P \end{aligned}$$

Using the multiplication property of determinants, we compute

$$\det(B - \lambda I) = \det[P^{-1}(A - \lambda I)P]$$

$$= \det(P^{-1}) \cdot \det(A - \lambda I) \cdot \det(P)$$

Since

$$\begin{aligned} \det(P^{-1}) \cdot \det(P) &= \det(P^{-1}P) \\ &= \det(I) = 1 \end{aligned}$$

we see that

$$\det(B - \lambda I) = \det(A - \lambda I)$$