PROBLEM:

Consider the vector space

$$R^3 = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} : x_1, x_2, x_3 \in R \right\}$$

and 2 bases of R^3:

$$\bar{e}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix},\ \bar{e}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix},\ \bar{e}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

and

$$\bar{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix},\ \bar{v}_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix},\ \bar{v}_3 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

Find coordinates of the vector

$$\bar{x} = (-4, 3, -5)$$

in $\{\bar{e}_1, \bar{e}_2, \bar{e}_3\}$ and in $\{\bar{v}_1, \bar{v}_2, \bar{v}_3\}$.

DEFINITION:

Suppose $\mathcal{B} = \{\bar{b}_1, \ldots, \bar{b}_n\}$ is a basis for a vector space V and $\bar{x}$ is in V. The coordinates of $\bar{x}$ relative to the basis $\mathcal{B}$ are the weights $c_1, \ldots, c_n$ such that

$$\bar{x} = c_1\bar{b}_1 + \ldots + c_n\bar{b}_n.$$

NOTATION:

$$[\bar{x}]_{\mathcal{B}} = \begin{bmatrix} c_1 \\ \ldots \\ c_n \end{bmatrix}$$

SOLUTION:

(a) Let

$$\mathcal{B}_1 = \{\bar{e}_1,\ \bar{e}_2,\ \bar{e}_3\}.$$

To find coordinates of the vector

$$\bar{x} = (-4, 3, -5)$$

relative to $\mathcal{B}_1$, we consider the augmented matrix

$$\begin{bmatrix} 1 & 0 & 0 & -4 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & -5 \end{bmatrix},$$

therefore

$$[\bar{x}]_{\mathcal{B}_1} = \begin{bmatrix} -4 \\ 3 \\ -5 \end{bmatrix}.$$

(b) Let

$$\mathcal{B}_2 = \{\bar{v}_1,\ \bar{v}_2,\ \bar{v}_3\}.$$

To find coordinates of the vector

$$\bar{x} = (-4, 3, -5)$$

relative to $\mathcal{B}_2$, we consider the augmented matrix

$$\begin{bmatrix} 1 & 1 & 0 & -4 \\ 1 & 0 & 1 & 3 \\ 0 & 1 & 1 & -5 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & -6 \\ 0 & 0 & 1 & 1 \end{bmatrix},$$

therefore

$$[\bar{x}]_{\mathcal{B}_2} = \begin{bmatrix} 2 \\ -6 \\ 1 \end{bmatrix}.$$

CONCLUSION:

The vector

$$\bar{x} = (-4, 3, -5)$$

has two different coordinates in two different bases:

$$[\bar{x}]_{\mathcal{B}_1} = \begin{bmatrix} -4 \\ 3 \\ -5 \end{bmatrix} \quad \text{and} \quad [\bar{x}]_{\mathcal{B}_2} = \begin{bmatrix} 2 \\ -6 \\ 1 \end{bmatrix},$$

where

$$\mathcal{B}_1 = \{\bar{e}_1,\ \bar{e}_2,\ \bar{e}_3\}$$

and

$$\mathcal{B}_2 = \{\bar{v}_1,\ \bar{v}_2,\ \bar{v}_3\}.$$

THEOREM:

Let $\mathcal{B} = \{\bar{b}_1, \ldots, \bar{b}_n\}$ and $\mathcal{C} = \{\bar{c}_1, \ldots, \bar{c}_n\}$ be bases of a vector space V. Then there is a unique matrix $\underset{\mathcal{C}\leftarrow\mathcal{B}}{P}$ such that

$$[\bar{x}]_{\mathcal{C}} = \underset{\mathcal{C}\leftarrow\mathcal{B}}{P}[\bar{x}]_{\mathcal{B}},$$

where

$$\underset{\mathcal{C}\leftarrow\mathcal{B}}{P} = [[\bar{b}_1]_{\mathcal{C}}\ \ [\bar{b}_2]_{\mathcal{C}}\ \ \ldots\ \ [\bar{b}_n]_{\mathcal{C}}].$$

EXAMPLE:

Let $\bar{x} = (-4, 3, -5)$, $C = \{\bar{e}_1,\ \bar{e}_2,\ \bar{e}_3\}$ and $B = \{\bar{v}_1,\ \bar{v}_2,\ \bar{v}_3\}$, then

$$[\bar{x}]_{\mathcal{C}} = \begin{bmatrix} -4 \\ 3 \\ -5 \end{bmatrix} \quad [\bar{x}]_{\mathcal{B}} = \begin{bmatrix} 2 \\ -6 \\ 1 \end{bmatrix},$$

$$[\bar{v}_1]_{\mathcal{C}} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix},\ [\bar{v}_2]_{\mathcal{C}} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix},\ [\bar{v}_3]_{\mathcal{C}} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

therefore

$$\underset{\mathcal{C}\leftarrow\mathcal{B}}{P} = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix},$$

and in fact

$$\begin{bmatrix} -4 \\ 3 \\ -5 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ -6 \\ 1 \end{bmatrix}$$

REMARK:

One can show that

$$\left(\underset{\mathcal{C}\leftarrow\mathcal{B}}{P}\right)^{-1} = \underset{\mathcal{B}\leftarrow\mathcal{C}}{P}$$

EXAMPLE:

We have $\underset{\mathcal{C}\leftarrow\mathcal{B}}{P} = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$, and

$$[\bar{x}]_{\mathcal{C}} = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} [\bar{x}]_{\mathcal{B}}$$

therefore

$$[\bar{x}]_{\mathcal{B}} = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}^{-1} [\bar{x}]_{\mathcal{C}},$$

so

$$\underset{\mathcal{B}\leftarrow\mathcal{C}}{P} = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}^{-1} = \left(\underset{\mathcal{C}\leftarrow\mathcal{B}}{P}\right)^{-1}.$$

PROBLEM:

Let $\mathcal{B} = \{\bar{b}_1, \bar{b}_2\}$ and $\mathcal{C} = \{\bar{c}_1, \bar{c}_2\}$ be bases for a vector space V, such that

$$\bar{b}_1 = 4\bar{c}_1 + \bar{c}_2$$

and

$$\bar{b}_2 = -6\bar{c}_1 + \bar{c}_2.$$

Suppose

$$\bar{x} = 3\bar{b}_1 + \bar{b}_2.$$

Find $[\bar{x}]_{\mathcal{C}}$.

SOLUTION:

We have $[\bar{x}]_{\mathcal{B}} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$ and

$$[\bar{b}_1]_{\mathcal{C}} = \begin{bmatrix} 4 \\ 1 \end{bmatrix}, \quad [\bar{b}_2]_{\mathcal{C}} = \begin{bmatrix} -6 \\ 1 \end{bmatrix},$$

therefore

$$\underset{\mathcal{C}\leftarrow\mathcal{B}}{P} = \begin{bmatrix} 4 & -6 \\ 1 & 1 \end{bmatrix},$$

hence

$$[\bar{x}]_{\mathcal{C}} = \begin{bmatrix} 4 & -6 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 6 \\ 4 \end{bmatrix}.$$

PROBLEM:

Let $\mathcal{B} = \{\bar{b}_1, \bar{b}_2\}$ and $\mathcal{C} = \{\bar{c}_1, \bar{c}_2\}$ be bases for a vector space V, such that

$$\bar{b}_1 = 6\bar{c}_1 - 2\bar{c}_2$$

and

$$\bar{b}_2 = 9\bar{c}_1 - 4\bar{c}_2.$$

Suppose

$$\bar{x} = -3\bar{b}_1 + 2\bar{b}_2.$$

Find $[\bar{x}]_{\mathcal{C}}$.

SOLUTION:

We have $[\bar{x}]_{\mathcal{B}} = \begin{bmatrix} -3 \\ 2 \end{bmatrix}$ and

$$[\bar{b}_1]_{\mathcal{C}} = \begin{bmatrix} 6 \\ -2 \end{bmatrix}, \quad [\bar{b}_2]_{\mathcal{C}} = \begin{bmatrix} 9 \\ -4 \end{bmatrix},$$

therefore

$$\underset{\mathcal{C}\leftarrow\mathcal{B}}{P} = \begin{bmatrix} 6 & 9 \\ -2 & -4 \end{bmatrix},$$

hence

$$[\bar{x}]_{\mathcal{C}} = \begin{bmatrix} 6 & 9 \\ -2 & -4 \end{bmatrix} \begin{bmatrix} -3 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ -2 \end{bmatrix}.$$

PROBLEM:

Let $\bar{b}_1 = \begin{bmatrix} 1 \\ -3 \end{bmatrix}$, $\bar{b}_2 = \begin{bmatrix} -2 \\ 4 \end{bmatrix}$, $\bar{c}_1 = \begin{bmatrix} -7 \\ 9 \end{bmatrix}$, $\bar{c}_2 = \begin{bmatrix} -5 \\ 7 \end{bmatrix}$, and consider the bases for R^2 given by $\mathcal{B} = \{\bar{b}_1, \bar{b}_2\}$ and $\mathcal{C} = \{\bar{c}_1, \bar{c}_2\}$.

(a) Find the change-of-coordinates matrix from $\mathcal{C}$ to $\mathcal{B}$.

(b) Find the change-of-coordinates matrix from $\mathcal{B}$ to $\mathcal{C}$.