DEFINITION:

The null space of an $m \times n$ matrix A, written as Nul A, is the set of all solutions to the homogeneous equation

$$A\bar{x} = \bar{0}.$$

DEFINITION:

The column space of an $m \times n$ matrix A, written as Col A, is the set of all linear combinations of the columns of A.

DEFINITION:

Let A be an $m \times n$ matrix. The row space is the set of all linear combinations of the row vectors of A.

EXAMPLE:

Let

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$

The row space of A is the subspace of R^4 spanned by

$$\bar{v}_1 = (1, 2, 3, 4)$$

$$\bar{v}_2 = (5, 6, 7, 8)$$

$$\bar{v}_3 = (0, 0, 1, 2)$$

THEOREM:

If two matrices A and B are row equivalent, then their row spaces are the same. If B is in echelon form, the nonzero rows of B form a basis for the row space of A as well as for that of B.

EXAMPLE:

Find a spanning set for the column space, row space, and null space of the matrix

$$A = \begin{bmatrix} -3 & 6 & -1 & 1 & -7 \\ 1 & -2 & 2 & 3 & -1 \\ 2 & -4 & 5 & 8 & -4 \end{bmatrix}.$$

SOLUTION:

(a) Obviously, columns of A, i.e.

$$\begin{bmatrix} -3 \\ 1 \\ 2 \end{bmatrix} \quad \begin{bmatrix} 6 \\ -2 \\ -4 \end{bmatrix} \quad \begin{bmatrix} -1 \\ 2 \\ 5 \end{bmatrix} \quad \begin{bmatrix} 1 \\ 3 \\ 8 \end{bmatrix} \quad \begin{bmatrix} -7 \\ -1 \\ -4 \end{bmatrix}$$

form the spanning set for Col A.

(b) Obviously, rows of A, i.e.

$$(-3,\ 6,\ -1,\ 1,\ -7)$$
$$(1,\ -2,\ 2,\ 3,\ -1)$$
$$(2,\ -4,\ 5,\ 8,\ -4)$$

form the spanning set for the row space of A.

(c) To find a spanning set for Nul A, we find the general solution of $A\bar{x} = \bar{0}$:

$$[A\ \bar{0}] \sim \begin{bmatrix} 1 & -2 & 0 & -1 & 3 & 0 \\ 0 & 0 & 1 & 2 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

therefore

$$\begin{cases} x_1 - 2x_2 - x_4 + 3x_5 = 0 \\ x_3 + 2x_4 - 2x_5 = 0, \end{cases}$$

so

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 2x_2 + x_4 - 3x_5 \\ x_2 \\ -2x_4 + 2x_5 \\ x_4 \\ x_5 \end{bmatrix} = x_2 \underbrace{\begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}}_{\bar{u}} + x_4 \underbrace{\begin{bmatrix} 1 \\ 0 \\ -2 \\ 1 \\ 0 \end{bmatrix}}_{\bar{v}} + x_5 \underbrace{\begin{bmatrix} -3 \\ 0 \\ 2 \\ 0 \\ 1 \end{bmatrix}}_{\bar{w}},$$

so Nul A =Span $\{\bar{u}, \bar{v}, \bar{w}\}$.

EXAMPLE:

Find bases for the row space, the column space, and the null space of the matrix

$$A = \begin{bmatrix} -1 & 4 & -2 & 0 & -3 \\ 2 & 1 & 1 & -1 & 0 \\ 0 & 9 & -3 & -1 & -6 \end{bmatrix}$$

SOLUTION:

Using elementary row operations, we get

$$\begin{bmatrix} -1 & 4 & -2 & 0 & -3 \\ 2 & 1 & 1 & -1 & 0 \\ 0 & 9 & -3 & -1 & -6 \end{bmatrix} \sim \begin{bmatrix} 1 & -4 & 2 & 0 & 3 \\ 0 & 9 & -3 & -1 & -6 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

(a) By the Theorem above, the first two rows of the second matrix form a basis for the row space of A.

(b) Since pivots are in columns 1 and 2, the first two columns of A form a basis for Col A.

(c) Finally, for Nul A we need the reduced echelon form. We have:

$$\begin{bmatrix} -1 & 4 & -2 & 0 & -3 \\ 2 & 1 & 1 & -1 & 0 \\ 0 & 9 & -3 & -1 & -6 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & \frac{2}{3} & -\frac{4}{9} & \frac{1}{3} \\ 0 & 1 & -\frac{1}{3} & -\frac{1}{9} & -\frac{2}{3} \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

therefore the corresponding system is

$$\begin{cases} x_1 + \frac{2}{3}x_3 - \frac{4}{9}x_4 + \frac{1}{3}x_5 = 0 \\ x_2 - \frac{1}{3}x_3 - \frac{1}{9}x_4 - \frac{2}{3}x_5 = 0 \end{cases}$$

Write the general solution in the parametric form

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} -\frac{2}{3}x_3 + \frac{4}{9}x_4 - \frac{1}{3}x_5 \\ \frac{1}{3}x_3 + \frac{1}{9}x_4 + \frac{2}{3}x_5 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix}$$

$$= x_3 \underbrace{\begin{bmatrix} -\frac{2}{3} \\ \frac{1}{3} \\ 1 \\ 0 \\ 0 \end{bmatrix}}_{\bar{v}_1} + x_4 \underbrace{\begin{bmatrix} \frac{4}{9} \\ \frac{1}{9} \\ 0 \\ 1 \\ 0 \end{bmatrix}}_{\bar{v}_2} + x_5 \underbrace{\begin{bmatrix} -\frac{1}{3} \\ \frac{2}{3} \\ 0 \\ 0 \\ 1 \end{bmatrix}}_{\bar{v}_3}$$

so $\{\bar{v}_1, \bar{v}_2, \bar{v}_3\}$ is the basis for Nul A.

DEFINITION:

The rank of A is the dimension of the column space of A.

EXAMPLE:

Since

$$A=\begin{bmatrix}1&2&0&-1\\2&0&1&-2\\4&4&-1&-4\\7&6&2&-7\end{bmatrix}\sim\begin{bmatrix}1&2&0&-1\\0&4&-1&0\\0&0&1&0\\0&0&0&0\end{bmatrix}$$

we have

$$\text{rank } A=3.$$

THEOREM (THE RANK THEOREM):

(a) The dimensions of the column space and the row space of an $m\times n$ matrix A are equal.

(b) This common dimension, the rank of A, also equals the number of pivot positions in A and satisfies the equation

$$\text{rank } A+\dim \text{Nul } A=n.$$

EXAMPLE:

Let

$$A=\begin{bmatrix}-1&4&-2&0&-3\\2&1&1&-1&0\\0&9&-3&-1&-6\end{bmatrix}$$

Using elementary row operations, we get

$$\begin{bmatrix}-1&4&-2&0&-3\\2&1&1&-1&0\\0&9&-3&-1&-6\end{bmatrix}\sim\begin{bmatrix}1&-4&2&0&3\\0&9&-3&-1&-6\\0&0&0&0&0\end{bmatrix}$$

(a) Since there are 2 pivots, we have

$$\dim \text{Row } A=\dim \text{Col } A=2.$$

(b) Since there are 3 free variables,

$$\dim \text{Nul } A=3.$$

We see that $2+3=5$ (# of columns).

EXAMPLE:

(a) If A is a 5×11 matrix with a 7-dimensional null space, what is the rank of A.

(b) Could a 5×11 matrix have a 5-dimensional null space?

SOLUTION:

(a) Since A has 11 columns, by the Theorem above we have

$$(\text{rank } A) + 7 = 11,$$

and hence rank $A = 4$

(b) No. If a 5×11 matrix had a 5-dimensional null space, it would have to have rank 6 by the Theorem above. But A has only 5 rows, therefore rank cannot exceed 5.

THEOREM:

Let A be a square $n \times n$ matrix. Then the following statements are equivalent:

(a) A is an invertible matrix.

(b) A is row equivalent to the $n \times n$ identity matrix.

(c) A has n pivot positions.

(d) The equation $A\bar{x} = \bar{0}$ has only the trivial solution.

(e) The columns of A form a linearly independent set.

(f) The equation $A\bar{x} = \bar{b}$ has at least one solution for each $\bar{b}$ in R^n.

(g) The columns of A span R^n.

(h) A^T is an invertible matrix.

(i) The columns of A form a basis of R^n.

(j) Col $A = R^n$

(k) dim Col $A = n$

(l) rank $A = n$

(m) Nul $A = \{\bar{0}\}$

(n) dim Nul $A = 0$