DEFINITION:

Let V be a vector space and B be a basis of V. The dimension of V is a number of vectors in B.

EXAMPLE:

(a) Since

$$\bar{e}_1 = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \ \bar{e}_2 = \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix}, \ldots, \ \bar{e}_n = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}$$

is the basis for R^n, we get $\dim R^n = n$.

(b) Since

$$\bar{e}_1 = 1, \ \bar{e}_2 = t, \ \bar{e}_3 = t^2, \ldots, \ \bar{e}_{n+1} = t^n$$

is the basis for P_n, we get $\dim P_n = n + 1$.

EXAMPLE:

Find the dimension of the subspace

$$H = \left\{ \begin{bmatrix} a - 4b + c \\ 2a - c + 3d \\ 2b - c + d \\ b + 3d \end{bmatrix} : a, b, c, d \in R \right\}$$

SOLUTION:

We have

$$\begin{bmatrix} a - 4b + c \\ 2a - c + 3d \\ 2b - c + 2d \\ b + 3d \end{bmatrix} = a \begin{bmatrix} 1 \\ 2 \\ 0 \\ 0 \end{bmatrix} + b \begin{bmatrix} -4 \\ 0 \\ 2 \\ 1 \end{bmatrix} + c \begin{bmatrix} 1 \\ -1 \\ -1 \\ 0 \end{bmatrix} + d \begin{bmatrix} 0 \\ 3 \\ 2 \\ 3 \end{bmatrix}$$

Using elementary row operations, we get

$$\begin{bmatrix} 1 & -4 & 1 & 0 \\ 2 & 0 & -1 & 3 \\ 0 & 2 & -1 & 2 \\ 0 & 1 & 0 & 3 \end{bmatrix} \sim \begin{bmatrix} 1 & -4 & 1 & 0 \\ 0 & 8 & -3 & 3 \\ 0 & 0 & 1 & -5 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

therefore $\dim H = 4$.

EXAMPLE:

Subspaces of R^3 can be classified by dimension:

0-dimensional subspaces: Only the zero subspace.

1-dimensional subspaces: Any subspace spanned by a single nonzero vector. Such subspaces are lines through the origin.

2-dimensional subspaces: Any subspace spanned by 2 linearly independent vectors (= not parallel). Such subspaces are planes through the origin.

3-dimensional subspaces: Only R^3 itself. Any 3 linearly independent vectors in R^3 (= not in the same plane) span all of R^3.

THEOREM:

Let V be a p-dimensional vector space, $p \geq 1$. Then

(a) Any linearly independent set of exactly p elements in V is automatically a basis for V.

(b) Any set of exactly p elements that spans V is automatically a basis for V.

THEOREM:

(a) The dimension of Nul A is the number of free variables in the equation $A\bar{x} = \bar{0}$.

(b) The dimension of Col A is the number of pivot columns in A.

EXAMPLE:

Find the dimensions of the null space and the column space of

$$A = \begin{bmatrix} 1 & 2 & 0 & -1 \\ 2 & 0 & 1 & -2 \\ 4 & 4 & -1 & -4 \\ 7 & 6 & 2 & -7 \end{bmatrix}$$

SOLUTION:

Using elementary row operations, we get

$$\begin{bmatrix} 1 & 2 & 0 & -1 \\ 2 & 0 & 1 & -2 \\ 4 & 4 & -1 & -4 \\ 7 & 6 & 2 & -7 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 0 & -1 \\ 0 & 4 & -1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

There is one free variable x_4. Hence dim Nul A=1. Also, dim Col $A = 3$ because A has 3 pivots.