DEFINITION:

Suppose $B = \{\bar{b}_1, \ldots, \bar{b}_n\}$ is a basis for a vector space V and $\bar{x}$ is in V. The coordinates of $\bar{x}$ relative to the basis B are the weights $c_1, \ldots, c_n$ such that

$$\bar{x} = c_1\bar{b}_1 + \ldots + c_n\bar{b}_n.$$

NOTATION:

$$[\bar{x}]_B = \begin{bmatrix} c_1 \\ \ldots \\ c_n \end{bmatrix}$$

THEOREM:

Let $B = \{\bar{b}_1, \ldots, \bar{b}_n\}$ be a basis for a vector space V. Then for each $\bar{x}$ in V, there exists a unique set of scalars $c_1, \ldots, c_n$ such that

$$\bar{x} = c_1\bar{b}_1 + \ldots + c_n\bar{b}_n.$$

EXAMPLE:

Let

$$\bar{b}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \ \bar{b}_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \ \bar{x} = \begin{bmatrix} 1 \\ 6 \end{bmatrix}.$$

Find coordinates of $\bar{x}$ in $\{\bar{b}_1, \ \bar{b}_2\}$.

SOLUTION:

We have

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 6 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 3 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 3 \end{bmatrix},$$

therefore

$$c_1 = -2 \quad \text{and} \quad c_2 = 3,$$

so

$$[\bar{x}]_B = \begin{bmatrix} -2 \\ 3 \end{bmatrix}.$$

PROBLEM:

Let $B = \{1, \ t, \ t^2\}$ be the standard basis for P_2. Find coordinates of the vector

$$\bar{p}(t) = -4 + 3t - 5t^2$$

relative to B.

SOLUTION:

By the definition above we have:

$$[\bar{p}]_B = \begin{bmatrix} -4 \\ 3 \\ -5 \end{bmatrix}.$$

PROBLEM:

Determine whether the polynomial

$$\bar{p}(t) = 2 + t + 7t^2 + 5t^3$$

can be represented as a linear combination of the polynomials

$$1 + t + 4t^2 + 3t^3, \quad 2 - t + 5t^2 + 3t^3.$$

SOLUTION:

The answer is "Yes", because

$$\begin{bmatrix} 1 & 2 & 2 \\ 1 & -1 & 1 \\ 4 & 5 & 7 \\ 3 & 3 & 5 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 2 \\ 0 & 3 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

and the echelon form of the augmented matrix represents a consistent system.

PROBLEM 1:

Determine whether the polynomials

$$1 + t^3, \quad 3 + t - 2t^2, \quad -t + 3t^2 - t^3$$

are linearly independent.

PROBLEM 2:

Determine whether the polynomials

$1-3t+5t^2$, $-3+5t-7t^2$, $-4+5t-6t^2$, $1-t^2$

span P_2.

PROBLEM 3:

Determine whether the polynomials

$3+7t$, $5+t-2t^3$, $t-2t^2$, $1+16t-6t^2+2t^3$

form a basis for P_3.

PROBLEM 4:

Determine whether the polynomials

$$1 + t, \ 1 + t^2, \ t + t^2$$

form a basis for P_2. If "Yes", find coordinates of the vector

$$\bar{p}(t) = -4 + 3t - 5t^2$$

relative to this basis.

Solution of Problem 1:

Let $B = \{1, \ t, \ t^2, \ t^3\}$ be the standard basis of P_3. Then polynomials

$$1 + t^3, \quad 3 + t - 2t^2, \quad -t + 3t^2 - t^3$$

produce coordinate vectors

$$\begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \quad \begin{bmatrix} 3 \\ 1 \\ -2 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} 0 \\ -1 \\ 3 \\ -1 \end{bmatrix}$$

relative to B. Writing these vectors as the columns of a matrix A, we can determine their independence:

$$\begin{bmatrix} 1 & 3 & 0 \\ 0 & 1 & -1 \\ 0 & -2 & 3 \\ 1 & 0 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

Since there are pivots in every column,

$$1 + t^3, \quad 3 + t - 2t^2, \quad -t + 3t^2 - t^3$$

are linearly independent.

Solution of Problem 2:

Let $B = \{1, \ t, \ t^2\}$ be the standard basis of P_2. Then polynomials

$1-3t+5t^2$, $-3+5t-7t^2$, $-4+5t-6t^2$, $1-t^2$

produce coordinate vectors

$$\begin{bmatrix} 1 \\ -3 \\ 5 \end{bmatrix}, \quad \begin{bmatrix} -3 \\ 5 \\ -7 \end{bmatrix}, \quad \begin{bmatrix} -4 \\ 5 \\ -6 \end{bmatrix}, \quad \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

relative to B. We have:

$$\begin{bmatrix} 1 & -3 & -4 & 1 \\ -3 & 5 & 5 & 0 \\ 5 & -7 & -6 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & -3 & -4 & 1 \\ 0 & 4 & 7 & -3 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Since there are 2 pivots and 3 rows, the polynomials

$1-3t+5t^2$, $-3+5t-7t^2$, $-4+5t-6t^2$, $1-t^2$

do not span P_2.

Solution of Problem 3:

Let $B = \{1, \ t, \ t^2, \ t^3\}$ be the standard basis of P_3. Then polynomials

$3+7t$, $5+t-2t^3$, $t-2t^2$, $1+16t-6t^2+2t^3$

produce coordinate vectors

$$\begin{bmatrix} 3 \\ 7 \\ 0 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} 5 \\ 1 \\ 0 \\ -2 \end{bmatrix}, \quad \begin{bmatrix} 0 \\ 1 \\ -2 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} 1 \\ 16 \\ -6 \\ 2 \end{bmatrix}$$

relative to B. We have:

$$\begin{bmatrix} 3 & 5 & 0 & 1 \\ 7 & 1 & 1 & 16 \\ 0 & 0 & -2 & -6 \\ 0 & -2 & 0 & 2 \end{bmatrix} \sim \begin{bmatrix} 3 & 5 & 0 & 1 \\ 0 & 32 & -3 & -41 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Since there are 3 pivots and 4 columns, the polynomials

$3+7t$, $5+t-2t^3$, $t-2t^2$, $1+16t-6t^2+2t^3$

do not form a basis for P_3.

Solution of Problem 4:

Let $B = \{1,\ t,\ t^2\}$ be the standard basis of P_2. Then polynomials

$$1+t,\ 1+t^2,\ t+t^2$$

produce coordinate vectors

$$\begin{bmatrix}1\\1\\0\end{bmatrix},\quad \begin{bmatrix}1\\0\\1\end{bmatrix},\quad \begin{bmatrix}0\\1\\1\end{bmatrix}$$

relative to B. We have:

$$\begin{bmatrix}1&1&0\\1&0&1\\0&1&1\end{bmatrix} \sim \begin{bmatrix}1&1&0\\0&-1&1\\0&0&2\end{bmatrix}.$$

Since there are 3 pivots and 3 columns, the polynomials

$$1+t,\ 1+t^2,\ t+t^2$$

form a basis for P_2.

Let

$$B = \{1+t,\ 1+t^2,\ t+t^2\}.$$

To find coordinates of the vector

$$\bar{p}(t) = -4+3t-5t^2$$

relative to B, we consider the augmented matrix

$$\begin{bmatrix}1&1&0&-4\\1&0&1&3\\0&1&1&-5\end{bmatrix} \sim \begin{bmatrix}1&0&0&2\\0&1&0&-6\\0&0&1&1\end{bmatrix},$$

therefore

$$[\bar{p}]_B = \begin{bmatrix}2\\-6\\1\end{bmatrix}.$$