DEFINITION:

The null space of an $m \times n$ matrix A, written as Nul A, is the set of all solutions to the homogeneous equation

$$A\bar{x} = \bar{0}.$$

DEFINITION$'$:

The null space of an $m \times n$ matrix A is the set of all $\bar{x}$ in R^n that are mapped into the zero vector $\bar{0}$ in R^m by the linear transformation

$$\bar{x} \longmapsto A\bar{x}.$$

EXAMPLE:

Let

$$A = \begin{bmatrix} 1 & -3 & -2 \\ -5 & 9 & 1 \end{bmatrix}.$$

Determine if $\bar{u} = \begin{bmatrix} 5 \\ 3 \\ -2 \end{bmatrix}$ belongs to the null space of A.

SOLUTION:

Since

$$A\bar{u} = \begin{bmatrix} 1 & -3 & -2 \\ -5 & 9 & 1 \end{bmatrix} \begin{bmatrix} 5 \\ 3 \\ -2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix},$$

$\bar{u}$ is in Nul A.

THEOREM:

The null space of an $m \times n$ matrix A is a subspace of R^n. Equivalently, the set of all solutions to a system $A\bar{x} = \bar{0}$ of m homogeneous linear equations in n unknowns is a subspace of R^n.

EXAMPLE:

Find a spanning set for the null space of the matrix

$$A = \begin{bmatrix} -3 & 6 & -1 & 1 & -7 \\ 1 & -2 & 2 & 3 & -1 \\ 2 & -4 & 5 & 8 & -4 \end{bmatrix}.$$

SOLUTION:

We find the general solution of $A\bar{x} = \bar{0}$:

$$[A\ \bar{0}] \sim \begin{bmatrix} 1 & -2 & 0 & -1 & 3 & 0 \\ 0 & 0 & 1 & 2 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

therefore

$$\begin{cases} x_1 - 2x_2 - x_4 + 3x_5 = 0 \\ x_3 + 2x_4 - 2x_5 = 0, \end{cases}$$

so

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 2x_2 + x_4 - 3x_5 \\ x_2 \\ -2x_4 + 2x_5 \\ x_4 \\ x_5 \end{bmatrix}$$

$$= x_2 \underbrace{\begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}}_{\bar{u}} + x_4 \underbrace{\begin{bmatrix} 1 \\ 0 \\ -2 \\ 1 \\ 0 \end{bmatrix}}_{\bar{v}} + x_5 \underbrace{\begin{bmatrix} -3 \\ 0 \\ 2 \\ 0 \\ 1 \end{bmatrix}}_{\bar{w}},$$

so Nul A =Span $\{\bar{u}, \bar{v}, \bar{w}\}$.

DEFINITION:

The column space of an $m \times n$ matrix A, written as Col A, is the set of all linear combinations of the columns of A.

REMARK:

So, if $A = [\bar{a}_1 \dots \bar{a}_n]$, then

$$\text{Col } A = \text{Span}\{\bar{a}_1, \dots, \bar{a}_n\}.$$

THEOREM:

The column space of an $m \times n$ matrix is a subspace of R^m.

EXAMPLE:

Let

$$A = \begin{bmatrix} 2 & 4 & -2 & 1 \\ -2 & -5 & 7 & 3 \\ 3 & 7 & -8 & 6 \end{bmatrix}.$$

Find a nonzero vector in Col A and a nonzero vector in Nul A.

SOLUTION:

1. Any column of A is a nonzero vector in Col A. For example, $\begin{bmatrix} 2 \\ -2 \\ 3 \end{bmatrix} =$

$$= 1 \begin{bmatrix} 2 \\ -2 \\ 3 \end{bmatrix} + 0 \begin{bmatrix} 4 \\ -5 \\ 7 \end{bmatrix} + 0 \begin{bmatrix} -2 \\ 7 \\ 8 \end{bmatrix} + 0 \begin{bmatrix} 1 \\ 3 \\ 6 \end{bmatrix}.$$

2. To find a nonzero vector in Nul A, we row reduce the augmented matrix $[A\ \bar{0}]$:

$$[A\ \bar{0}] \sim \begin{bmatrix} 1 & 0 & 9 & 0 & 0 \\ 0 & 1 & -5 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix},$$

therefore any vector

$$\bar{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -9x_3 \\ 5x_3 \\ x_3 \\ 0 \end{bmatrix}$$

is in Nul A. For example, if we put $x_3 = 1$, we get

$$\bar{u} = \begin{bmatrix} -9 \\ 5 \\ 1 \\ 0 \end{bmatrix}$$

is in Nul A.