DEFINITION:

A vector space is a nonempty set V of objects, called vectors, on which are defined two operations, called addition and multiplication by scalars (real numbers), subject to the following 10 axioms (or rules):

1. The sum of $\bar{u}$ and $\bar{v}$, denoted by $\bar{u}+\bar{v}$, is in V.
2. $\bar{u}+\bar{v}=\bar{v}+\bar{u}$.
3. $(\bar{u}+\bar{v})+\bar{w}=\bar{u}+(\bar{v}+\bar{w})$.
4. There is a zero vector $\bar{0}$ in V such that $\bar{u}+\bar{0}=\bar{u}$.
5. For each $\bar{u}$ in V, there is a vector $-\bar{u}$ in V such that $\bar{u}+(-\bar{u})=\bar{0}$.
6. The scalar multiple of $\bar{u}$ by c, denoted by $c\bar{u}$, is in V.
7. $c(\bar{u}+\bar{v})=c\bar{u}+c\bar{v}$.
8. $(c+d)\bar{u}=c\bar{u}+d\bar{u}$.
9. $c(d\bar{u})=(cd)\bar{u}$.
10. $1\cdot\bar{u}=\bar{u}$.

These axioms must hold for all vectors $\bar{u}$, $\bar{v}$, and $\bar{w}$ in V and all scalars c and d.

EXAMPLE:

R^n is a vector space. In fact, let

$$\bar{u}=\begin{bmatrix}x_1\\x_2\\\vdots\\x_n\end{bmatrix}\qquad \bar{v}=\begin{bmatrix}y_1\\y_2\\\vdots\\y_n\end{bmatrix}\qquad \bar{w}=\begin{bmatrix}z_1\\z_2\\\vdots\\z_n\end{bmatrix}$$

Then

1. $\bar{u}+\bar{v}$ is in V.
2. $\bar{u}+\bar{v}=\bar{v}+\bar{u}$.
3. $(\bar{u}+\bar{v})+\bar{w}=\bar{u}+(\bar{v}+\bar{w})$.
4. There is the zero vector $\bar{0}=\begin{bmatrix}0\\0\\\vdots\\0\end{bmatrix}$ in V such that $\bar{u}+\bar{0}=\bar{u}$, since

$$\bar{u}+\bar{0}=\begin{bmatrix}x_1\\x_2\\\vdots\\x_n\end{bmatrix}+\begin{bmatrix}0\\0\\\vdots\\0\end{bmatrix}=\begin{bmatrix}x_1\\x_2\\\vdots\\x_n\end{bmatrix}=\bar{u}.$$

5. For each $\bar{u}=\begin{bmatrix}x_1\\x_2\\\vdots\\x_n\end{bmatrix}$ in V, there is the vector $-\bar{u}=\begin{bmatrix}-x_1\\-x_2\\\vdots\\-x_n\end{bmatrix}$ in V such that $\bar{u}+(-\bar{u})=\bar{0}$, since

$$\bar{u}+(-\bar{u})=\begin{bmatrix}x_1\\x_2\\\vdots\\x_n\end{bmatrix}+\begin{bmatrix}-x_1\\-x_2\\\vdots\\-x_n\end{bmatrix}=\begin{bmatrix}0\\0\\\vdots\\0\end{bmatrix}=\bar{0}.$$

6. The scalar multiple of $\bar{u}$ by c, denoted by $c\bar{u}$, is in V.
7. $c(\bar{u}+\bar{v})=c\bar{u}+c\bar{v}$.
8. $(c+d)\bar{u}=c\bar{u}+d\bar{u}$.
9. $c(d\bar{u})=(cd)\bar{u}$.
10. $1\cdot\bar{u}=\bar{u}$.

EXAMPLE:

The set of all $n\times m$ matrices, i.e.

$$\begin{bmatrix}x_{11}&\dots&x_{1m}\\x_{21}&\dots&x_{2m}\\\dots&&\\x_{n1}&\dots&x_{nm}\end{bmatrix}$$

Then

1. $\bar{u}+\bar{v}$ is in V.
2. $\bar{u}+\bar{v}=\bar{v}+\bar{u}$.
3. $(\bar{u}+\bar{v})+\bar{w}=\bar{u}+(\bar{v}+\bar{w})$.
4. There is the zero vector $\bar{0}=\begin{bmatrix}0&\dots&0\\0&\dots&0\\\dots&&\\0&\dots&0\end{bmatrix}$ in V such that $\bar{u}+\bar{0}=\bar{u}$.

5. For each $\bar{u} = \begin{bmatrix} x_{11} & \dots & x_{1m} \\ x_{21} & \dots & x_{2m} \\ \dots \\ x_{n1} & \dots & x_{nm} \end{bmatrix}$ in V,

there is the vector $-\bar{u} = \begin{bmatrix} -x_{11} & \dots & -x_{1m} \\ -x_{21} & \dots & -x_{2m} \\ \dots \\ -x_{n1} & \dots & -x_{nm} \end{bmatrix}$ in V such that $\bar{u} + (-\bar{u}) = \bar{0}$.

6. The scalar multiple of $\bar{u}$ by c, denoted by $c\bar{u}$, is in V.
7. $c(\bar{u} + \bar{v}) = c\bar{u} + c\bar{v}$.
8. $(c + d)\bar{u} = c\bar{u} + d\bar{u}$.
9. $c(d\bar{u}) = (cd)\bar{u}$.
10. $1 \cdot \bar{u} = \bar{u}$.

EXAMPLE:

The set P_n of polynomials of degree at most n:

$$\bar{p}(t) = a_n t^n + \ldots + a_2 t^2 + a_1 t + a_0$$

where the coefficients $a_n, \ldots, a_0$ and the variable t are real numbers.

EXAMPLE:

The set of all real-valued functions defined on R.

DEFINITION:
A subspace of a vector space V is a subset H of V that has 3 properties:

1. The zero vector of V is in H.
2. H is closed under vector addition. That is, for each $\bar{u}$ and $\bar{v}$ in H, the sum $\bar{u} + \bar{v}$ is in H.
3. H is closed under multiplication by scalars. That is, for each $\bar{u}$ in H and each scalar c, the vector $c\bar{u}$ is in H.

REMARK:

One can show that a subspace H of a vector space V is a vector space.

EXAMPLE:

The set consisting of only the zero vector $\bar{0}$ in a vector space V is a subspace of V, called the zero subspace and written as $\{\bar{0}\}$.

EXAMPLE:

Given $\bar{v}_1$ and $\bar{v}_2$ in a vector space V, let $H = \text{Span}\ \{\bar{v}_1, \bar{v}_2\}$. Show that H is a subspace of V.

SOLUTION:

First of all, note that

Span $\{\bar{v}_1, \bar{v}_2\} = \{\alpha\bar{v}_1 + \beta\bar{v}_2 : \alpha, \beta \in R\}$.

Therefore Span $\{\bar{v}_1, \bar{v}_2\}$ is a subset of V. Moreover,

1. The zero vector $\bar{0}$ is in H, since

$$\bar{0} = 0 \cdot \bar{v}_1 + 0 \cdot \bar{v}_2.$$

2. H is closed under vector addition. In fact, let

$$\bar{u} = s_1\bar{v}_1 + s_2\bar{v}_2, \quad \bar{w} = t_1\bar{v}_1 + t_2\bar{v}_2.$$

By Axioms 2, 3, and 8 we have:

$$\bar{u} + \bar{w} = (s_1\bar{v}_1 + s_2\bar{v}_2) + (t_1\bar{v}_1 + t_2\bar{v}_2)$$
$$(s_1 + t_1)\bar{v}_1 + (s_2 + t_2)\bar{v}_2,$$

therefore $\bar{u} + \bar{w}$ is in H.

3. Similarly, if c is any scalar, then by Axioms 7 and 9 we get

$$c\bar{u} = c(s_1\bar{v}_1 + s_2\bar{v}_2) = (cs_1)\bar{v}_1 + (cs_2)\bar{v}_2,$$

therefore $c\bar{u}$ is also in H.
Thus, H is a subspace of V.

THEOREM:

If $\bar{v}_1, \ldots, \bar{v}_p$ are in a vector space V, then Span $\{\bar{v}_1, \ldots, \bar{v}_p\}$ is a subspace of V.