DEFINITION:
A vector space is a nonempty set \( V \) of objects, called vectors, on which are defined two operations, called addition and multiplication by scalars (real numbers), subject to the following 10 axioms (or rules):

1. The sum of \( \bar{u} \) and \( \bar{v} \), denoted by \( \bar{u} + \bar{v} \), is in \( V \).
2. \( \bar{u} + \bar{v} = \bar{v} + \bar{u} \).
3. \( \bar{u} + (\bar{v} + \bar{w}) = \bar{u} + (\bar{v} + \bar{w}) \).
4. There is a zero vector \( \bar{0} \) in \( V \) such that \( \bar{u} + \bar{0} = \bar{u} \).
5. For each \( \bar{u} \) in \( V \), there is a vector \( -\bar{u} \) in \( V \) such that \( \bar{u} + (-\bar{u}) = \bar{0} \).
6. The scalar multiple of \( \bar{u} \) by \( c \), denoted by \( c\bar{u} \), is in \( V \).
7. \( c(\bar{u} + \bar{v}) = c\bar{u} + c\bar{v} \).
8. \( (c + d)\bar{u} = c\bar{u} + d\bar{u} \).
9. \( c(d\bar{u}) = (cd)\bar{u} \).
10. \( 1 \cdot \bar{u} = \bar{u} \).

These axioms must hold for all vectors \( \bar{u}, \bar{v}, \) and \( \bar{w} \) in \( V \) and all scalars \( c \) and \( d \).

EXAMPLE:
\( \mathbb{R}^n \) is a vector space. In fact, let
\[
\bar{u} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, \quad \bar{v} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}, \quad \bar{w} = \begin{bmatrix} z_1 \\ z_2 \\ \vdots \\ z_n \end{bmatrix}
\]
Then
1. \( \bar{u} + \bar{v} \) is in \( V \).
2. \( \bar{u} + \bar{v} = \bar{v} + \bar{u} \).
3. \( \bar{u} + \bar{v} + \bar{w} = \bar{u} + (\bar{v} + \bar{w}) \).
4. There is the zero vector \( \bar{0} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \) in \( V \) such that \( \bar{u} + \bar{0} = \bar{u} \), since
\[
\bar{u} + \bar{0} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \bar{u}.
\]

EXAMPLE:
The set of all \( n \times m \) matrices, i.e.
\[
\begin{bmatrix}
  x_{11} & \cdots & x_{1m} \\
  x_{21} & \cdots & x_{2m} \\
  \vdots & & \vdots \\
  x_{n1} & \cdots & x_{nm}
\end{bmatrix}
\]
Then
1. \( \bar{u} + \bar{v} \) is in \( V \).
2. \( \bar{u} + \bar{v} = \bar{v} + \bar{u} \).
3. \( \bar{u} + \bar{v} + \bar{w} = \bar{u} + (\bar{v} + \bar{w}) \).
4. There is the zero vector \( \bar{0} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \) in \( V \) such that \( \bar{u} + \bar{0} = \bar{u} \).
5. For each \( \bar{u} = \begin{bmatrix} x_{11} & \cdots & x_{1m} \\ x_{21} & \cdots & x_{2m} \\ \vdots \\ x_{n1} & \cdots & x_{nm} \end{bmatrix} \) in \( V \), there is the vector \( -\bar{u} = \begin{bmatrix} -x_{11} & \cdots & -x_{1m} \\ -x_{21} & \cdots & -x_{2m} \\ \vdots \\ -x_{n1} & \cdots & -x_{nm} \end{bmatrix} \) in \( V \) such that \( \bar{u} + (-\bar{u}) = \bar{0} \).

6. The scalar multiple of \( \bar{u} \) by \( c \), denoted by \( c\bar{u} \), is in \( V \).

7. \( c(\bar{u} + \bar{v}) = c\bar{u} + c\bar{v} \).

8. \( (c + d)\bar{u} = c\bar{u} + d\bar{u} \).

9. \( c(d\bar{u}) = (cd)\bar{u} \).

10. \( 1 \cdot \bar{u} = \bar{u} \).

**EXAMPLE:**

The set \( P_n \) of polynomials of degree at most \( n \): \( \bar{p}(t) = a_n t^n + \ldots + a_2 t^2 + a_1 t + a_0 \) where the coefficients \( a_n, \ldots, a_0 \) and the variable \( t \) are real numbers.

**EXAMPLE:**

The set consisting of all real-valued functions defined on \( R \).

**DEFINITION:**

A subspace of a vector space \( V \) is a subset \( H \) of \( V \) that has 3 properties:

1. The zero vector of \( V \) is in \( H \).
2. \( H \) is closed under vector addition. That is, for each \( \bar{u} \) and \( \bar{v} \) in \( H \), the sum \( \bar{u} + \bar{v} \) is in \( H \).
3. \( H \) is closed under multiplication by scalars. That is, for each \( \bar{u} \) in \( H \) and each scalar \( c \), the vector \( c\bar{u} \) is in \( H \).

**REMARK:**

One can show that a subspace \( H \) of a vector space \( V \) is a vector space.
SOLUTION:

First of all, note that
Span \( \{ \bar{v}_1, \bar{v}_2 \} = \{ \alpha \bar{v}_1 + \beta \bar{v}_2 : \alpha, \beta \in R \} \).
Therefore Span \( \{ \bar{v}_1, \bar{v}_2 \} \) is a subset of \( V \).
Moreover,
1. The zero vector \( \bar{0} \) is in \( H \), since
   \( \bar{0} = 0 \cdot \bar{v}_1 + 0 \cdot \bar{v}_2 \).
2. \( H \) is closed under vector addition.
   In fact, let
   \( \bar{u} = s_1 \bar{v}_1 + s_2 \bar{v}_2, \quad \bar{w} = t_1 \bar{v}_1 + t_2 \bar{v}_2 \).
By Axioms 2, 3, and 8 we have:
   \( \bar{u} + \bar{w} = (s_1 \bar{v}_1 + s_2 \bar{v}_2) + (t_1 \bar{v}_1 + t_2 \bar{v}_2) \)
   \( = (s_1 + t_1) \bar{v}_1 + (s_2 + t_2) \bar{v}_2 \),
therefore \( \bar{u} + \bar{w} \) is in \( H \).
3. Similarly, if \( c \) is any scalar, then by Axioms 7 and 9 we get
   \( c \bar{u} = c(s_1 \bar{v}_1 + s_2 \bar{v}_2) = (cs_1) \bar{v}_1 + (cs_2) \bar{v}_2 \),
therefore \( c \bar{u} \) is also in \( H \).
Thus, \( H \) is a subspace of \( V \).

THEOREM:
If \( \bar{v}_1, \ldots, \bar{v}_p \) are in a vector space \( V \), then
Span \( \{ \bar{v}_1, \ldots, \bar{v}_p \} \) is a subspace of \( V \).