CRAMER'S RULE

DEFINITION:

For any $n \times n$ matrix A and any $\bar{b}$ in R^n, let $A_i(\bar{b})$ be the matrix obtained from A by replacing column i by the vector $\bar{b}$.

EXAMPLE:

Let $A = \begin{bmatrix} 2 & 1 & 3 \\ 1 & 0 & 4 \\ 0 & 0 & 5 \end{bmatrix}$, $\bar{b} = \begin{bmatrix} 3 \\ 8 \\ 9 \end{bmatrix}$. Then

$$A_1(\bar{b}) = \begin{bmatrix} 3 & 1 & 3 \\ 8 & 0 & 4 \\ 9 & 0 & 5 \end{bmatrix} \qquad A_2(\bar{b}) = \begin{bmatrix} 2 & 3 & 3 \\ 1 & 8 & 4 \\ 0 & 9 & 5 \end{bmatrix}$$

$$A_3(\bar{b}) = \begin{bmatrix} 2 & 1 & 3 \\ 1 & 0 & 8 \\ 0 & 0 & 9 \end{bmatrix}$$

THEOREM (CRAMER'S RULE):

Let A be an invertible $n \times n$ matrix. For any $\bar{b}$ in R^n, the unique solution $\bar{x}$ of $A\bar{x} = \bar{b}$ has entries given by

$$x_i = \frac{\det A_i(\bar{b})}{\det A}, \quad i = 1, 2, \ldots, n.$$

PROBLEM: Solve using Cramer's rule

$$\begin{cases} x_1 - 2x_2 = 1 \\ 3x_1 + 4x_2 = -7 \end{cases}$$

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SOLUTION: We have

$$x_1 = \frac{\begin{vmatrix} 1 & -2 \\ -7 & 4 \end{vmatrix}}{\begin{vmatrix} 1 & -2 \\ 3 & 4 \end{vmatrix}} = \frac{4 - 14}{4 - (-6)} = \frac{-10}{10} = -1$$

$$x_2 = \frac{\begin{vmatrix} 1 & 1 \\ 3 & -7 \end{vmatrix}}{\begin{vmatrix} 1 & -2 \\ 3 & 4 \end{vmatrix}} = \frac{-7 - 3}{10} = \frac{-10}{10} = -1$$

FORMULA FOR A^{-1}

DEFINITION:

For any $n \times n$ matrix A, let A_{ij} be the submatrix of A, formed by deleting row i and column j.

EXAMPLE: Let $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 0 \end{bmatrix}$. Then

$$A_{11} = \begin{bmatrix} 5 & 6 \\ 8 & 0 \end{bmatrix} \quad A_{12} = \begin{bmatrix} 4 & 6 \\ 7 & 0 \end{bmatrix} \quad A_{13} = \begin{bmatrix} 4 & 5 \\ 7 & 8 \end{bmatrix}$$

$$A_{21} = \begin{bmatrix} 2 & 3 \\ 8 & 0 \end{bmatrix} \quad A_{22} = \begin{bmatrix} 1 & 3 \\ 7 & 0 \end{bmatrix} \quad A_{23} = \begin{bmatrix} 1 & 2 \\ 7 & 8 \end{bmatrix}$$

$$A_{31} = \begin{bmatrix} 2 & 3 \\ 5 & 6 \end{bmatrix} \quad A_{32} = \begin{bmatrix} 1 & 3 \\ 4 & 6 \end{bmatrix} \quad A_{33} = \begin{bmatrix} 1 & 2 \\ 4 & 5 \end{bmatrix}$$

THEOREM (AN INVERSE FORMULA): Let A be an invertible $n \times n$ matrix. Then

$$A^{-1} = \frac{1}{\det A} \begin{bmatrix} C_{11} & C_{12} & \dots & C_{1n} \\ C_{21} & C_{22} & \dots & C_{2n} \\ \dots & \dots & \dots & \dots \\ C_{n1} & C_{n2} & \dots & C_{nn} \end{bmatrix}^T,$$

where

$$C_{ij} = (-1)^{i+j} \det A_{ij}.$$

EXAMPLE: Let $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 0 \end{bmatrix}$. Find A^{-1}.

SOLUTION:

Step 1: One can verify that $\det A = 27$.

Step 2: We have

$$A_{11} = \begin{bmatrix} 5 & 6 \\ 8 & 0 \end{bmatrix} \quad A_{12} = \begin{bmatrix} 4 & 6 \\ 7 & 0 \end{bmatrix} \quad A_{13} = \begin{bmatrix} 4 & 5 \\ 7 & 8 \end{bmatrix}$$

$$\det A_{11} = -48 \quad \det A_{12} = -42 \quad \det A_{13} = -3$$

$$C_{11} = -48 \quad C_{12} = 42 \quad C_{13} = -3$$

$$A_{21} = \begin{bmatrix} 2 & 3 \\ 8 & 0 \end{bmatrix} \quad A_{22} = \begin{bmatrix} 1 & 3 \\ 7 & 0 \end{bmatrix} \quad A_{23} = \begin{bmatrix} 1 & 2 \\ 7 & 8 \end{bmatrix}$$

$$\det A_{21} = -24 \quad \det A_{22} = -21 \quad \det A_{23} = -6$$

$$C_{21} = 24 \quad C_{22} = -21 \quad C_{23} = 6$$

$$A_{31} = \begin{bmatrix} 2 & 3 \\ 5 & 6 \end{bmatrix} \quad A_{32} = \begin{bmatrix} 1 & 3 \\ 4 & 6 \end{bmatrix} \quad A_{33} = \begin{bmatrix} 1 & 2 \\ 4 & 5 \end{bmatrix}$$

$$\det A_{31} = -3 \quad \det A_{32} = -6 \quad \det A_{33} = -3$$

$$C_{31} = -3 \quad C_{32} = 6 \quad C_{33} = -3$$

Step 3:

$$A^{-1} = \frac{1}{27} \begin{bmatrix} -48 & 42 & -3 \\ 24 & -21 & 6 \\ -3 & 6 & -3 \end{bmatrix}^T = \frac{1}{27} \begin{bmatrix} -48 & 24 & -3 \\ 42 & -21 & 6 \\ -3 & 6 & -3 \end{bmatrix}$$

$$= \begin{bmatrix} -16/9 & 8/9 & -1/9 \\ 14/9 & -7/9 & 2/9 \\ -1/9 & 2/9 & -1/9 \end{bmatrix}$$

AREA AND VOLUME

THEOREM:

If A is a 2×2 matrix, the area of the parallelogram determined by the columns of A is $|\det A|$. If A is a 3×3 matrix, the volume of the parallelepiped determined by the columns of A is $|\det A|$.

THEOREM:

Let $T : R^2 \to R^2$ be the linear transformation determined by a 2×2 matrix A. If S is a parallelogram in R^2, then

$$\{\text{area of } T(S)\} = |\det A| \cdot \{\text{area of } S\}$$

If T is determined by a 3×3 matrix A, and if S is a parallelepiped in R^3, then

$$\{\text{volume of } T(S)\} = |\det A| \cdot \{\text{volume of } S\}$$