THEOREM:

Let A be a square matrix.

(a) If a multiple of one row (column) of A is added to another row (column) to produce a matrix B, then $\det A = \det B$.

EXAMPLE: $\begin{vmatrix} 1 & 1 & 2 \\ 1 & 2 & 3 \\ 3 & 3 & 8 \end{vmatrix} = \begin{vmatrix} 1 & 1 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 2 \end{vmatrix} = 2.$

(b) If two rows (columns) of A are interchanged to produce B, then $\det A = -\det B$.

EXAMPLE: $\begin{vmatrix} 1 & 1 & 2 \\ 1 & 2 & 3 \\ 3 & 3 & 8 \end{vmatrix} = -\begin{vmatrix} 1 & 2 & 3 \\ 1 & 1 & 2 \\ 3 & 3 & 8 \end{vmatrix}$

(c) If one row (column) of A is multiplied by k to produce B, then $\det B = k \det A$.

EXAMPLE: $\begin{vmatrix} 100 & 300 \\ 1 & 2 \end{vmatrix} = 100 \begin{vmatrix} 1 & 3 \\ 1 & 2 \end{vmatrix}.$

PROBLEM: Find

$$\begin{vmatrix} 1 & 3 & 5 & 4 \\ 2 & -3 & 1 & -1 \\ -1 & 2 & -1 & 0 \\ 2 & 2 & 5 & 3 \end{vmatrix}$$

SOLUTION: We have

$$\begin{vmatrix} 1 & 3 & 5 & 4 \\ 2 & -3 & 1 & -1 \\ -1 & 2 & -1 & 0 \\ 2 & 2 & 5 & 3 \end{vmatrix} = \begin{vmatrix} 1 & 3 & 5 & 4 \\ 0 & -9 & -9 & -9 \\ 0 & 5 & 4 & 4 \\ 0 & -4 & -5 & -5 \end{vmatrix}$$

$$= (-9) \begin{vmatrix} 1 & 3 & 5 & 4 \\ 0 & 1 & 1 & 1 \\ 0 & 5 & 4 & 4 \\ 0 & -4 & -5 & -5 \end{vmatrix}$$

$$= (-9) \begin{vmatrix} 1 & 3 & 5 & 4 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & -1 & -1 \\ 0 & 0 & -1 & -1 \end{vmatrix}$$

Since the last two rows are equal, the determinant is equal to 0.

THEOREM:

A square matrix A is invertible if and only if $\det A \neq 0$.

THEOREM:

Let A be a square matrix. Then

(a) $\det A^T = \det A$.

(b) $\det(AB) = \det A \det B$.

(c) $\det(A^{-1}) = (\det A)^{-1}$.