PROBLEM: Solve

$$\begin{cases} 3x_1 - 5x_2 = 7 \\ 8x_1 + 9x_2 = -5 \end{cases}$$

SOLUTION: We have

$$x_1 = \frac{\begin{vmatrix} 7 & -5 \\ -5 & 9 \end{vmatrix}}{\begin{vmatrix} 3 & -5 \\ 8 & 9 \end{vmatrix}} = \frac{63 - 25}{27 + 40} = \frac{38}{67}$$

$$x_2 = \frac{\begin{vmatrix} 3 & 7 \\ 8 & -5 \end{vmatrix}}{\begin{vmatrix} 3 & -5 \\ 8 & 9 \end{vmatrix}} = \frac{-15 - 56}{67} = -\frac{71}{67}$$

PROBLEM: Solve

$$\begin{cases} x_1 - x_2 + x_3 = 1 \\ x_1 + x_2 - x_3 = 2 \\ x_1 - x_2 - x_3 = 3 \end{cases}$$

SOLUTION: We have

$$x_1 = \frac{\begin{vmatrix} 1 & -1 & 1 \\ 2 & 1 & -1 \\ 3 & -1 & -1 \end{vmatrix}}{\begin{vmatrix} 1 & -1 & 1 \\ 1 & 1 & -1 \\ 1 & -1 & -1 \end{vmatrix}} =???$$

Let

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{bmatrix}$$

DEFINITION:

The determinant of an $n \times n$ matrix A is the following sum:

$$\det A = a_{11} \det A_{11} - a_{12} \det A_{12} + a_{13} \det A_{13} - \dots + (-1)^{n+1} a_{1n} \det A_{1n},$$

where A_{1j} are submatrices formed by deleting from A the first row and jth column.

PROBLEM: Find $\begin{vmatrix} 1 & 0 & 0 & 1 \\ 0 & 2 & 0 & -1 \\ -1 & 0 & 3 & 0 \\ 0 & 1 & 1 & 1 \end{vmatrix}$.

SOLUTION: We have

$$\begin{vmatrix} 1 & 0 & 0 & 1 \\ 0 & 2 & 0 & -1 \\ -1 & 0 & 3 & 0 \\ 0 & 1 & 1 & 1 \end{vmatrix} = \begin{vmatrix} 2 & 0 & -1 \\ 0 & 3 & 0 \\ 1 & 1 & 1 \end{vmatrix} - \begin{vmatrix} 0 & 2 & 0 \\ -1 & 0 & 3 \\ 0 & 1 & 1 \end{vmatrix}$$

Since

$$\begin{vmatrix} 2 & 0 & -1 \\ 0 & 3 & 0 \\ 1 & 1 & 1 \end{vmatrix} = 2 \begin{vmatrix} 3 & 0 \\ 1 & 1 \end{vmatrix} + (-1) \begin{vmatrix} 0 & 3 \\ 1 & 1 \end{vmatrix} = 2(3 - 0) + (-1)(0 - 3) = 9$$

and

$$\begin{vmatrix} 0 & 2 & 0 \\ -1 & 0 & 3 \\ 0 & 1 & 1 \end{vmatrix} = -2 \begin{vmatrix} -1 & 3 \\ 0 & 1 \end{vmatrix} = -2(-1) = 2$$

it follows that the determinant is equal to $9 - 2 = 7$.

THEOREM:

If A is a triangular matrix, then det A is the product of the entries on the main diagonal of A.

EXAMPLE:

$$\begin{vmatrix} 3 & -7 & 8 & 9 & -6 \\ 0 & 2 & -5 & 7 & 3 \\ 0 & 0 & 1 & 5 & 0 \\ 0 & 0 & 0 & 4 & -1 \\ 0 & 0 & 0 & 0 & 1 \end{vmatrix} = 3 \cdot 2 \cdot 1 \cdot 4 \cdot 1 = 24$$

THEOREM:

We have det $A = 0$

(a) if A contains a zero-row or zero-column.

EXAMPLE: $\begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 0 & 0 & 0 \end{vmatrix} = 0$

(b) if A contains two similar rows or columns.

EXAMPLE: $\begin{vmatrix} 1 & 2 & 1 \\ 2 & 5 & 2 \\ 2 & 0 & 2 \end{vmatrix} = 0$

(c) if some row (column) of A is a multiple of some other row (column) of A.

EXAMPLE: $\begin{vmatrix} 1 & 2 & 1 \\ -2 & -4 & -2 \\ 3 & 5 & 7 \end{vmatrix} = 0$