DEFINITION:

Suppose $B = \{\bar{b}_1, \ldots, \bar{b}_n\}$ is a basis for a subspace H of R^n and $\bar{x}$ is in H. The coordinates of $\bar{x}$ relative to the basis B are the weights $c_1, \ldots, c_n$ such that

$$\bar{x} = c_1\bar{b}_1 + \ldots + c_n\bar{b}_n.$$

NOTATION:

$$[\bar{x}]_B = \begin{bmatrix} c_1 \\ \cdots \\ c_n \end{bmatrix}$$

THEOREM:

Let $B = \{\bar{b}_1, \ldots, \bar{b}_n\}$ be a basis for a subspace H of R^n. Then for each $\bar{x}$ in H, there exists a unique set of scalars $c_1, \ldots, c_n$ such that

$$\bar{x} = c_1\bar{b}_1 + \ldots + c_n\bar{b}_n.$$

EXAMPLE:

Let

$$\bar{b}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \ \bar{b}_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \ \bar{x} = \begin{bmatrix} 1 \\ 6 \end{bmatrix}.$$

Find coordinates of $\bar{x}$ in $\{\bar{b}_1, \ \bar{b}_2\}$.

SOLUTION:

We have

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 6 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 3 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 3 \end{bmatrix},$$

therefore

$$c_1 = -2 \quad \text{and} \quad c_2 = 3,$$

so

$$[\bar{x}]_B = \begin{bmatrix} -2 \\ 3 \end{bmatrix}.$$

DEFINITION:

Let H be a subspace of R^n and B be a basis of H. The dimension of H is a number of vectors in B.

EXAMPLE:

Since

$$\bar{e}_1 = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \ \bar{e}_2 = \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix}, \ldots, \ \bar{e}_n = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}$$

is the basis for R^n, we get $\dim R^n = n$.

EXAMPLE:

Find the dimension of the subspace

$$H = \left\{ \begin{bmatrix} a - 4b + c \\ 2a - c + 3d \\ 2b - c + d \\ b + 3d \end{bmatrix} : a, b, c, d \in R \right\}$$

SOLUTION:

We have

$$\begin{bmatrix} a-4b+c \\ 2a-c+3d \\ 2b-c+2d \\ b+3d \end{bmatrix} = a\begin{bmatrix} 1 \\ 2 \\ 0 \\ 0 \end{bmatrix} + b\begin{bmatrix} -4 \\ 0 \\ 2 \\ 1 \end{bmatrix} + c\begin{bmatrix} 1 \\ -1 \\ -1 \\ 0 \end{bmatrix} + d\begin{bmatrix} 0 \\ 3 \\ 2 \\ 3 \end{bmatrix}$$

Using elementary row operations, we get

$$\begin{bmatrix} 1 & -4 & 1 & 0 \\ 2 & 0 & -1 & 3 \\ 0 & 2 & -1 & 2 \\ 0 & 1 & 0 & 3 \end{bmatrix} \sim \begin{bmatrix} 1 & -4 & 1 & 0 \\ 0 & 8 & -3 & 3 \\ 0 & 0 & 1 & -5 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

therefore dim $H = 4$.

EXAMPLE:

Subspaces of R^3 can be classified by dimension:

0-dimensional subspaces: Only the zero subspace.

1-dimensional subspaces: Any subspace spanned by a single nonzero vector. Such subspaces are lines through the origin.

2-dimensional subspaces: Any subspace spanned by 2 linearly independent vectors (= not parallel). Such subspaces are planes through the origin.

3-dimensional subspaces: Only R^3 itself. Any 3 linearly independent vectors in R^3 (= not in the same plane) span all of R^3.

THEOREM:

(a) The dimension of Nul A is the number of free variables in the equation $A\bar{x} = \bar{0}$.

(b) The dimension of Col A is the number of pivot columns in A.

EXAMPLE:

Find the dimensions of the null space and the column space of

$$A = \begin{bmatrix} 1 & 2 & 0 & -1 \\ 2 & 0 & 1 & -2 \\ 4 & 4 & -1 & -4 \\ 7 & 6 & 2 & -7 \end{bmatrix}$$

SOLUTION:

Using elementary row operations, we get

$$\begin{bmatrix} 1 & 2 & 0 & -1 \\ 2 & 0 & 1 & -2 \\ 4 & 4 & -1 & -4 \\ 7 & 6 & 2 & -7 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 0 & -1 \\ 0 & 4 & -1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

There is one free variable x_4. Hence dim Nul A=1. Also, dim Col $A = 3$ because A has 3 pivots.

DEFINITION:

The rank of A is the dimension of the column space of A.

EXAMPLE:

Since

$$A = \begin{bmatrix} 1 & 2 & 0 & -1 \\ 2 & 0 & 1 & -2 \\ 4 & 4 & -1 & -4 \\ 7 & 6 & 2 & -7 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 0 & -1 \\ 0 & 4 & -1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

we have

$$\text{rank } A = 3.$$

THEOREM (THE RANK THEOREM):

If a matrix A has n columns, then

$$\text{rank } A + \dim \text{Nul } A = n.$$

EXAMPLE:

Let

$$A = \begin{bmatrix} -1 & 4 & -2 & 0 & -3 \\ 2 & 1 & 1 & -1 & 0 \\ 0 & 9 & -3 & -1 & -6 \end{bmatrix}$$

Using elementary row operations, we get

$$\begin{bmatrix} -1 & 4 & -2 & 0 & -3 \\ 2 & 1 & 1 & -1 & 0 \\ 0 & 9 & -3 & -1 & -6 \end{bmatrix} \sim \begin{bmatrix} 1 & -4 & 2 & 0 & 3 \\ 0 & 9 & -3 & -1 & -6 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Since there are 2 pivots, we have

$$\text{rank } A = 2$$

Since there are 3 free variables,

$$\dim \text{Nul } A = 3.$$

We see that $2 + 3 = 5$ (# of columns).

EXAMPLE:

(a) If A is a 5×11 matrix with a 7-dimensional null space, what is the rank of A.

(b) Could a 5×11 matrix have a 5-dimensional null space?

SOLUTION:

(a) Since A has 11 columns, by the Theorem above we have

$$(\text{rank } A) + 7 = 11,$$

and hence rank $A = 4$

(b) No. If a 5×11 matrix had a 5-dimensional null space, it would have to have rank 6 by the Theorem above. But A has only 5 rows, therefore rank cannot exceed 5.

THEOREM:

Let A be a square $n \times n$ matrix. Then the following statements are equivalent:

(a) A is an invertible matrix.

(b) A is row equivalent to the $n \times n$ identity matrix.

(c) A has n pivot positions.

(d) The equation $A\bar{x} = \bar{0}$ has only the trivial solution.

(e) The columns of A form a linearly independent set.

(f) The equation $A\bar{x} = \bar{b}$ has at least one solution for each $\bar{b}$ in R^n.

(g) The columns of A span R^n.

(h) A^T is an invertible matrix.

(i) The columns of A form a basis of R^n.

(j) Col $A = R^n$

(k) dim Col $A = n$

(l) rank $A = n$

(m) Nul $A = \{\bar{0}\}$

(n) dim Nul $A = 0$