NOTATION:

$$R^n = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} : x_1, \dots, x_n \in R \right\}$$

DEFINITION:
A subspace of R^n is a subset H of R^n that has 3 properties:

1. The zero vector is in H.
2. H is closed under vector addition. That is, for each $\bar{u}$ and $\bar{v}$ in H, the sum $\bar{u} + \bar{v}$ is in H.
3. H is closed under multiplication by scalars. That is, for each $\bar{u}$ in H and each scalar c, the vector $c\bar{u}$ is in H.

WARNING:

R^2 is not a subspace of R^3, because R^2 is not a subset of R^3.

EXAMPLE:

The set consisting of only the zero vector $\bar{0}$ is a subspace of R^n, called the zero subspace and written as $\{\bar{0}\}$.

EXAMPLE:

The set

$$H = \left\{ \begin{bmatrix} s \\ t \\ 0 \end{bmatrix} : s \text{ and } t \text{ are real numbers} \right\}$$

is a subspace of R^3.

THEOREM:

If $\bar{v}_1, \dots, \bar{v}_p$ are in a vector space R^n, then Span $\{\bar{v}_1, \dots, \bar{v}_p\}$ is a subspace of R^n.

EXAMPLE:

Let

$$\bar{v}_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \quad \bar{v}_2 = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}.$$

By the Theorem above

$$\text{Span}\{\bar{v}_1, \bar{v}_2\}$$

is a subspace of R^3.

EXAMPLE:

Let H be the set of all vectors of the form

$$\begin{bmatrix} 4a - b \\ 2b \\ a - 2b \\ a - b \end{bmatrix}$$

where a and b are arbitrary scalars. Show that H is a subspace of R^4.

SOLUTION:

We have

$$\begin{bmatrix} 4a-b \\ 2b \\ a-2b \\ a-b \end{bmatrix} = a \underbrace{\begin{bmatrix} 4 \\ 0 \\ 1 \\ 1 \end{bmatrix}}_{\bar{v}_1} + b \underbrace{\begin{bmatrix} -1 \\ 2 \\ -2 \\ -1 \end{bmatrix}}_{\bar{v}_2}$$

We see that

$$H = \mathrm{Span}\{\bar{v}_1, \bar{v}_2\}$$

therefore H is a subspace of R^4 by the Theorem above.

EXAMPLE:

Let H be the set of all vectors of the form

$$\begin{bmatrix} a-b \\ b-c \\ c-a \\ b \end{bmatrix}$$

where a, b and c are arbitrary scalars. Find a set S of vectors that spans H or show that H is not a vector space.

SOLUTION:

We have

$$\begin{bmatrix} a-b \\ b-c \\ c-a \\ b \end{bmatrix} = a \underbrace{\begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \end{bmatrix}}_{\bar{v}_1} + b \underbrace{\begin{bmatrix} -1 \\ 1 \\ 0 \\ 1 \end{bmatrix}}_{\bar{v}_2} + c \underbrace{\begin{bmatrix} 0 \\ -1 \\ 1 \\ 0 \end{bmatrix}}_{\bar{v}_3}$$

and we see that H is a vector space and

$$\{\bar{v}_1, \bar{v}_2, \bar{v}_3\}$$

spans H.

EXAMPLE:

Let H be the set of all vectors of the form

$$\begin{bmatrix} 3a+b \\ 4 \\ a-5b \end{bmatrix}$$

where a and b are arbitrary scalars. Show that H is not a vector space.

SOLUTION:

H is not a vector space, since $\bar{0} \notin H$ (the second entry is always nonzero).

DEFINITION:

The null space of an $m \times n$ matrix A, written as Nul A, is the set of all solutions to the homogeneous equation

$$A\bar{x} = \bar{0}.$$

DEFINITION′:

The null space of an $m \times n$ matrix A is the set of all $\bar{x}$ in R^n that are mapped into the zero vector $\bar{0}$ in R^m by the linear transformation

$$\bar{x} \longmapsto A\bar{x}.$$

EXAMPLE:

Let

$$A = \begin{bmatrix} 1 & -2 & -1 \\ 2 & -3 & -4 \end{bmatrix}.$$

Determine if $\bar{u} = \begin{bmatrix} 5 \\ 2 \\ 1 \end{bmatrix}$ belongs to the null space of A.

SOLUTION:

Since

$$A\bar{u} = \begin{bmatrix} 1 & -2 & -1 \\ 2 & -3 & -4 \end{bmatrix} \begin{bmatrix} 5 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix},$$

$\bar{u}$ is in Nul A.

EXAMPLE:

Let

$$A = \begin{bmatrix} 1 & -3 & -2 \\ -5 & 9 & 0 \end{bmatrix}.$$

Determine if $\bar{u} = \begin{bmatrix} 5 \\ 3 \\ -2 \end{bmatrix}$ belongs to the null space of A.

SOLUTION:

Since

$$A\bar{u} = \begin{bmatrix} 1 & -3 & -2 \\ -5 & 9 & 0 \end{bmatrix} \begin{bmatrix} 5 \\ 3 \\ -2 \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \end{bmatrix},$$

$\bar{u}$ is not in Nul A.

THEOREM:

The null space of an $m \times n$ matrix A is a subspace of R^n. Equivalently, the set of all solutions to a system $A\bar{x} = \bar{0}$ of m homogeneous linear equations in n unknowns is a subspace of R^n.

EXAMPLE:

Find a spanning set for the null space of the matrix

$$A = \begin{bmatrix} -3 & 6 & -1 & 1 & -7 \\ 1 & -2 & 2 & 3 & -1 \\ 2 & -4 & 5 & 8 & -4 \end{bmatrix}.$$

SOLUTION:

We find the general solution of $A\bar{x} = \bar{0}$:

$$[A\ \bar{0}] \sim \begin{bmatrix} 1 & -2 & 0 & -1 & 3 & 0 \\ 0 & 0 & 1 & 2 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

therefore

$$\begin{cases} x_1 - 2x_2 - x_4 + 3x_5 = 0 \\ x_3 + 2x_4 - 2x_5 = 0, \end{cases}$$

so $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 2x_2 + x_4 - 3x_5 \\ x_2 \\ -2x_4 + 2x_5 \\ x_4 \\ x_5 \end{bmatrix}$

$$= x_2 \underbrace{\begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}}_{\bar{u}} + x_4 \underbrace{\begin{bmatrix} 1 \\ 0 \\ -2 \\ 1 \\ 0 \end{bmatrix}}_{\bar{v}} + x_5 \underbrace{\begin{bmatrix} -3 \\ 0 \\ 2 \\ 0 \\ 1 \end{bmatrix}}_{\bar{w}},$$

so Nul A =Span $\{\bar{u}, \bar{v}, \bar{w}\}$.

DEFINITION:

The column space of an $m \times n$ matrix A, written as Col A, is the set of all linear combinations of the columns of A.

REMARK:

So, if $A = [\bar{a}_1 \ldots \bar{a}_n]$, then

$$\text{Col } A = \text{Span}\{\bar{a}_1, \ldots, \bar{a}_n\}.$$

THEOREM:

The column space of an $m \times n$ matrix is a subspace of R^m.

EXAMPLE:

Let

$$A = \begin{bmatrix} 2 & 4 & -2 & 1 \\ -2 & -5 & 7 & 3 \\ 3 & 7 & -8 & 6 \end{bmatrix}.$$

Find a nonzero vector in Col A and a nonzero vector in Nul A.

SOLUTION:

1. Any column of A is a nonzero vector in Col A. For example, $\begin{bmatrix} 2 \\ -2 \\ 3 \end{bmatrix} =$

$$= 1 \begin{bmatrix} 2 \\ -2 \\ 3 \end{bmatrix} + 0 \begin{bmatrix} 4 \\ -5 \\ 7 \end{bmatrix} + 0 \begin{bmatrix} -2 \\ 7 \\ 8 \end{bmatrix} + 0 \begin{bmatrix} 1 \\ 3 \\ 6 \end{bmatrix}.$$

2. To find a nonzero vector in Nul A, we row reduce the augmented matrix $[A\ \bar{0}]$:

$$[A\ \bar{0}] \sim \begin{bmatrix} 1 & 0 & 9 & 0 & 0 \\ 0 & 1 & -5 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix},$$

therefore any vector

$$\bar{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -9x_3 \\ 5x_3 \\ x_3 \\ 0 \end{bmatrix}$$

is in Nul A. For example, if we put $x_3 = 1$, we get

$$\bar{u} = \begin{bmatrix} -9 \\ 5 \\ 1 \\ 0 \end{bmatrix}$$

is in Nul A.

<u>DEFINITION</u>:

Let H be a subspace of a vector space R^n. A set of vectors

$$B = \{\bar{b}_1, \ldots, \bar{b}_p\}$$

in R^n is a <u>basis</u> for H if

(a) B is a linearly independent set;

(b) $H = \text{Span}\ \{\bar{b}_1, \ldots, \bar{b}_p\}$.

<u>REMARK</u>:

In other words, a <u>basis</u> for H is a minimal number of vectors which span H.

<u>EXAMPLE</u>:

Let

$$\bar{e}_1 = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \ \bar{e}_2 = \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix}, \ldots, \bar{e}_n = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}$$

The set $\{\bar{e}_1, \ldots, \bar{e}_n\}$ is a basis for R^n, because

(a) they are <u>linearly independent</u>, since

(# of columns) = (# or pivots)

(b) they <u>span R^n</u>, since

there are n pivots.

<u>DEFINITION:</u>

The set $\{\bar{e}_1, \ldots, \bar{e}_n\}$ is called the standard basis for R^n.

<u>THEOREM</u>:

The set of vectors $\{\bar{v}_1, \ldots, \bar{v}_p\}$ is a basis of R^n if and only if $n = p$ and the matrix $A = [\bar{v}_1 \ldots \bar{v}_p]$ has exactly n pivot positions.

<u>PROBLEM</u>:

Let

$$\bar{v}_1 = \begin{bmatrix} 3 \\ 0 \\ -6 \end{bmatrix}, \ \bar{v}_2 = \begin{bmatrix} -4 \\ 1 \\ 7 \end{bmatrix}, \ \bar{v}_3 = \begin{bmatrix} -2 \\ 1 \\ 5 \end{bmatrix}.$$

Determine if $\{\bar{v}_1,\ \bar{v}_2,\ \bar{v}_3\}$ is a basis for R^3.

SOLUTION:

We have

$$\begin{bmatrix} 3 & -4 & -2 \\ 0 & 1 & 1 \\ -6 & 7 & 5 \end{bmatrix} \sim \begin{bmatrix} 3 & -4 & -2 \\ 0 & 1 & 1 \\ 0 & -1 & 1 \end{bmatrix} \sim \begin{bmatrix} 3 & -4 & -2 \\ 0 & 1 & 1 \\ 0 & 0 & 2 \end{bmatrix}.$$

Since we have 3 vectors and 3 pivots, $\{\bar{v}_1,\ \bar{v}_2,\ \bar{v}_3\}$ is a basis for R^3.

THEOREM:

The pivot columns of a matrix A form a basis for Col A.

PROBLEM:

Let

$$\bar{v}_1 = \begin{bmatrix} 3 \\ 0 \\ -6 \end{bmatrix},\ \bar{v}_2 = \begin{bmatrix} -4 \\ 1 \\ 7 \end{bmatrix},\ \bar{v}_3 = \begin{bmatrix} -2 \\ 1 \\ 3 \end{bmatrix}.$$

Find a basis for Col $[\bar{v}_1\ \bar{v}_2\ \bar{v}_3]$.

SOLUTION:

We have

$$\begin{bmatrix} 3 & -4 & -2 \\ 0 & 1 & 1 \\ -6 & 7 & 3 \end{bmatrix} \sim \begin{bmatrix} 3 & -4 & -2 \\ 0 & 1 & 1 \\ 0 & -1 & -1 \end{bmatrix} \sim \begin{bmatrix} 3 & -4 & -2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

Since the first and the second columns are pivot columns, $\{\bar{v}_1,\ \bar{v}_2\}$ is a basis for Col $[\bar{v}_1\ \bar{v}_2\ \bar{v}_3]$.

PROBLEM:

It can be shown that the matrix

$$\begin{bmatrix} 1 & 4 & 0 & 2 & -1 \\ 3 & 12 & 1 & 5 & 5 \\ 2 & 8 & 1 & 3 & 2 \\ 5 & 20 & 2 & 8 & 8 \end{bmatrix}$$

is row equivalent to the matrix

$$\begin{bmatrix} 1 & 4 & 0 & 2 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Find the bases for Col A and Nul A.

SOLUTION:

(a) By the Theorem above, $\{\bar{v}_1,\ \bar{v}_3,\ \bar{v}_5\}$ is a basis for Col A.

(b) To find the basis for Nul A, we consider a system

$$\begin{cases} x_1 + 4x_2 + 2x_4 = 0 \\ x_3 - x_4 = 0 \\ x_5 = 0. \end{cases}$$

Write the general solution in the parametric form

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} -4x_2 - 2x_4 \\ x_2 \\ x_4 \\ x_4 \\ 0 \end{bmatrix} = x_2 \underbrace{\begin{bmatrix} -4 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}}_{\bar{v}_1} + x_4 \underbrace{\begin{bmatrix} -2 \\ 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}}_{\bar{v}_2}$$

so $\{\bar{v}_1, \bar{v}_2\}$ is the basis for Nul A.

EXAMPLE:

Find bases for the column space and the null space of the matrix

$$A = \begin{bmatrix} -1 & 4 & -2 & 0 & -3 \\ 2 & 1 & 1 & -1 & 0 \\ 0 & 9 & -3 & -1 & -6 \end{bmatrix}$$

SOLUTION:

Using elementary row operations, we get

$$\begin{bmatrix} -1 & 4 & -2 & 0 & -3 \\ 2 & 1 & 1 & -1 & 0 \\ 0 & 9 & -3 & -1 & -6 \end{bmatrix} \sim \begin{bmatrix} 1 & -4 & 2 & 0 & 3 \\ 0 & 9 & -3 & -1 & -6 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

(a) Since pivots are in columns 1 and 2, the first two columns of A form a basis for Col A.

(b) For Nul A we need the reduced echelon form. We have:

$$\begin{bmatrix} -1 & 4 & -2 & 0 & -3 \\ 2 & 1 & 1 & -1 & 0 \\ 0 & 9 & -3 & -1 & -6 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & \frac{2}{3} & -\frac{4}{9} & \frac{1}{3} \\ 0 & 1 & -\frac{1}{3} & -\frac{1}{9} & -\frac{2}{3} \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

therefore the corresponding system is

$$\begin{cases} x_1 + \frac{2}{3}x_3 - \frac{4}{9}x_4 + \frac{1}{3}x_5 = 0 \\ x_2 - \frac{1}{3}x_3 - \frac{1}{9}x_4 - \frac{2}{3}x_5 = 0 \end{cases}$$

Write the general solution in the parametric form

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} -\frac{2}{3}x_3 + \frac{4}{9}x_4 - \frac{1}{3}x_5 \\ \frac{1}{3}x_3 + \frac{1}{9}x_4 + \frac{2}{3}x_5 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix}$$

$$= x_3 \underbrace{\begin{bmatrix} -\frac{2}{3} \\ \frac{1}{3} \\ 1 \\ 0 \\ 0 \end{bmatrix}}_{\bar{v}_1} + x_4 \underbrace{\begin{bmatrix} \frac{4}{9} \\ \frac{1}{9} \\ 0 \\ 1 \\ 0 \end{bmatrix}}_{\bar{v}_2} + x_5 \underbrace{\begin{bmatrix} -\frac{1}{3} \\ \frac{2}{3} \\ 0 \\ 0 \\ 1 \end{bmatrix}}_{\bar{v}_3}$$

so $\{\bar{v}_1, \bar{v}_2, \bar{v}_3\}$ is the basis for Nul A.

TEST 1:

Vectors $\bar{v}_1, \ldots, \bar{v}_p$ are linearly independent if and only if the matrix $A = [\bar{v}_1 \ldots \bar{v}_p]$ has p pivots.

TEST 2:

Vectors $\bar{v}_1, \ldots, \bar{v}_p$ span R^n if and only if the matrix $A = [\bar{v}_1 \ldots \bar{v}_p]$ has n pivots.

TEST 3:

Vectors $\bar{v}_1, \ldots, \bar{v}_p$ form a basis of R^n if and only if the matrix $A = [\bar{v}_1 \ldots \bar{v}_p]$ has n pivots and $p = n$.