DEFINITION:

If A and B are $m\times n$ matrices, then the sum $A+B$ is the $m\times n$ matrix whose entries are the sums of the corresponding entries of A and B.

EXAMPLE:

$$\begin{bmatrix} 1 & -2 & -1 \\ -2 & 3 & -2 \end{bmatrix} + \begin{bmatrix} 0 & 1 & 2 \\ -2 & -4 & -1 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 1 \\ -4 & -1 & -3 \end{bmatrix}$$

REMARK: We can add matrices only of the same size.

EXAMPLE:

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} + \begin{bmatrix} 1 & 2 \\ 1 & 0 \\ 3 & 4 \end{bmatrix} = ???$$

DEFINITION:

If r is a scalar and A is a matrix, then the scalar multiple rA is the matrix whose entries are r times the corresponding entries in A.

EXAMPLE:

$$(-2)\begin{bmatrix} 1 & 2 & -3 \\ -1 & 0 & -2 \end{bmatrix} = \begin{bmatrix} -2 & -4 & 6 \\ 2 & 0 & 4 \end{bmatrix}$$

PROPERTIES:

Let A, B, and C be matrices of the same size, and let r and s be scalars. Then

(a) $A+B=B+A$

(b) $(A+B)+C=A+(B+C)$

(c) $r(A+B)=rA+rB$

(d) $(r+s)A=rA+sA$

(e) $r(sA)=(rs)A$

DEFINITION:

If A is an $m\times n$ matrix, and if B is an $n\times p$ matrix with columns $\bar{b}_1,\dots,\bar{b}_p$, then the product AB is the $m\times p$ matrix whose columns are $A\bar{b}_1,\dots,A\bar{b}_p$. That is,

$$\begin{aligned} AB &= A[\bar{b}_1 \ \ \bar{b}_2 \ \ \dots \ \ \bar{b}_p] \\ &= [A\bar{b}_1 \ \ A\bar{b}_2 \ \ \dots \ \ A\bar{b}_p] \end{aligned}$$

ROW-COLUMN RULE FOR COMPUTING AB:

If the product AB is defined, then the entry in row i and column j of AB is the sum of the products of corresponding entries from row i of A and column j of B. If $(AB)_{ij}$ denotes the (i,j)-entry in AB, and if A is an $m\times n$ matrix, then

$$(AB)_{ij}=a_{i1}b_{1j}+a_{i2}b_{2j}+\ldots+a_{in}b_{nj}$$

EXAMPLE:

Let $A = \begin{bmatrix} 2 & 3 \\ 1 & -5 \end{bmatrix}$ and $B = \begin{bmatrix} 4 & 3 & 6 \\ 1 & -2 & 3 \end{bmatrix}$, then

AB

$$= \begin{bmatrix} 2\cdot4+3\cdot1 & 2\cdot3+3\cdot(-2) & 2\cdot6+3\cdot3 \\ 1\cdot4+(-5)\cdot1 & 1\cdot3+(-5)\cdot(-2) & 1\cdot6+(-5)\cdot3 \end{bmatrix}$$

$$= \begin{bmatrix} 11 & 0 & 21 \\ -1 & 13 & -9 \end{bmatrix}$$

Note that BA is undefined.

PROBLEM:

Consider the following matrices:

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & 4 \end{bmatrix}, B = \begin{bmatrix} 1 & 0 \\ 2 & 1 \\ 3 & 2 \end{bmatrix}, C = \begin{bmatrix} 3 & -2 \\ 2 & 5 \end{bmatrix}.$$

If possible, compute:

(a) AB

(b) $AC + B^2$

(c) $AB + C^2$

SOLUTION:

We have:

(a) $AB = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & 4 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 2 & 1 \\ 3 & 2 \end{bmatrix} = \begin{bmatrix} 14 & 8 \\ 16 & 9 \end{bmatrix}$.

(b) Impossible.

(c) $AB + C^2$

$$= \begin{bmatrix} 14 & 8 \\ 16 & 9 \end{bmatrix} + \begin{bmatrix} 3 & -2 \\ 2 & 5 \end{bmatrix} \begin{bmatrix} 3 & -2 \\ 2 & 5 \end{bmatrix}$$

$$= \begin{bmatrix} 14 & 8 \\ 16 & 9 \end{bmatrix} + \begin{bmatrix} 5 & -16 \\ 16 & 21 \end{bmatrix}$$

$$= \begin{bmatrix} 19 & -8 \\ 32 & 30 \end{bmatrix}.$$

PROPERTIES:

Let A be an $m \times n$ matrix, and let B and C have sizes for which the indicated sums and products are defined. Then

(a) $A(BC) = (AB)C$

(b) $A(B + C) = AB + AC$

(c) $(B + C)A = BA + CA$

(d) $r(AB) = (rA)B = A(rB)$

WARNING

1. In general, $AB \neq BA$.

EXAMPLE:

Let $A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 1 \\ 2 & 0 \end{bmatrix}$. Then

$$AB = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 2 & 0 \end{bmatrix} = \begin{bmatrix} 5 & 1 \\ 2 & 0 \end{bmatrix}$$

$$BA = \begin{bmatrix} 1 & 1 \\ 2 & 0 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}$$

So,

$$AB \neq BA.$$

EXAMPLE:

Let $A = \begin{bmatrix} 1 & 2 & 3 \end{bmatrix}$ and $B = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}$, then

$$AB = \begin{bmatrix} 1 \cdot 4 + 2 \cdot 5 + 3 \cdot 6 \end{bmatrix} = [32]$$

and

$$BA = \begin{bmatrix} 4 \cdot 1 & 4 \cdot 2 & 4 \cdot 3 \\ 5 \cdot 1 & 5 \cdot 2 & 5 \cdot 3 \\ 6 \cdot 1 & 6 \cdot 2 & 6 \cdot 3 \end{bmatrix}$$

$$= \begin{bmatrix} 4 & 8 & 12 \\ 5 & 10 & 15 \\ 6 & 12 & 18 \end{bmatrix}$$

WARNING

2. If $AB = AC$, then it is not true in general that $B = C$.

EXAMPLE: Let

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, B = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \text{and } C = \begin{bmatrix} 0 & 0 \\ 0 & 2 \end{bmatrix}.$$

Then

$$AB = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$AC = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

So,

$$AB = AC, \quad \text{but} \quad B \neq C.$$

WARNING

3. If $AB = 0$, then it is not true in general that $A = 0$ or $B = 0$.

EXAMPLE:

Let $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$. Then

$$AB = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

So,

$$AB = 0, \quad \text{but} \quad A \neq 0 \quad \text{and} \quad B \neq 0.$$

THE TRANSPOSE OF A MATRIX

DEFINITION:

Let A be an $m \times n$ matrix. The transpose of A is the $n \times m$ matrix, denoted by A^T, whose columns are formed from the corresponding rows of A.

EXAMPLE:

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \qquad A^T = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}$$

$$B = \begin{bmatrix} -3 & 1 \\ 4 & 7 \\ 8 & -5 \end{bmatrix} \qquad B^T = \begin{bmatrix} -3 & 4 & 8 \\ 1 & 7 & -5 \end{bmatrix}$$

$$C = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \end{bmatrix} \qquad C^T = \begin{bmatrix} 1 & 5 \\ 2 & 6 \\ 3 & 7 \\ 4 & 8 \end{bmatrix}$$

PROPERTIES:

Let A and B denote matrices whose sizes are appropriate for the following sums and products. Then

(a) $(A^T)^T = A$

(b) $(A + B)^T = A^T + B^T$

(c) $(rA)^T = rA^T$ for any scalar r

(d) $(AB)^T = B^T A^T$