PROBLEM:

Let $\bar{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ and

$$A_1 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \quad A_2 = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$A_3 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \quad A_4 = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$$

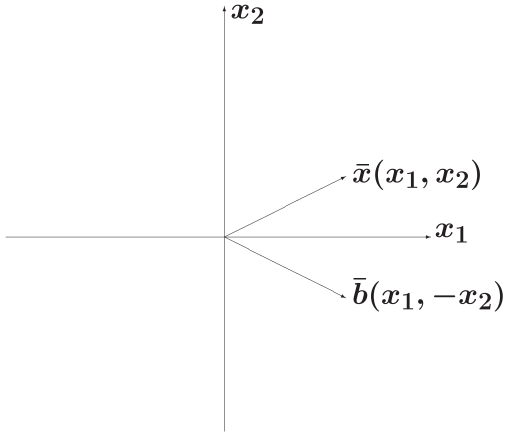
$$A_5 = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$

$$B_1 = \begin{bmatrix} k & 0 \\ 0 & 1 \end{bmatrix} \quad B_2 = \begin{bmatrix} 1 & 0 \\ 0 & k \end{bmatrix}$$

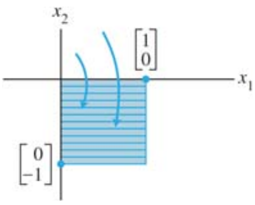
$$C_1 = \begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix} \quad C_2 = \begin{bmatrix} 1 & 0 \\ k & 1 \end{bmatrix}$$

Find $A_i\bar{x}$, $B_i\bar{x}$, $C_i\bar{x}$. Provide illustrations and geometric explanations.

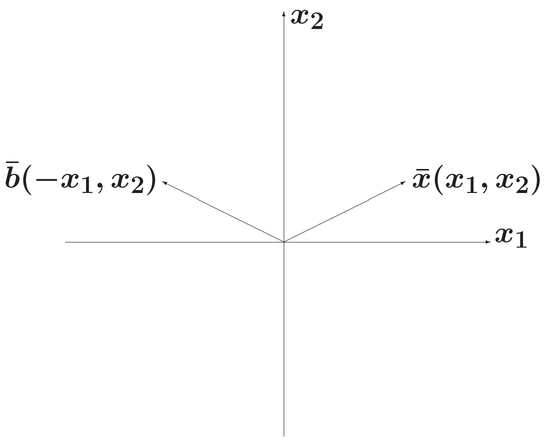
1. $A_1\bar{x} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ -x_2 \end{bmatrix}$



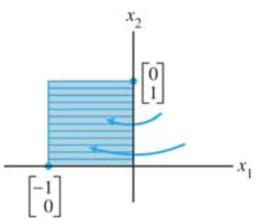
Reflection through the x_1-axis



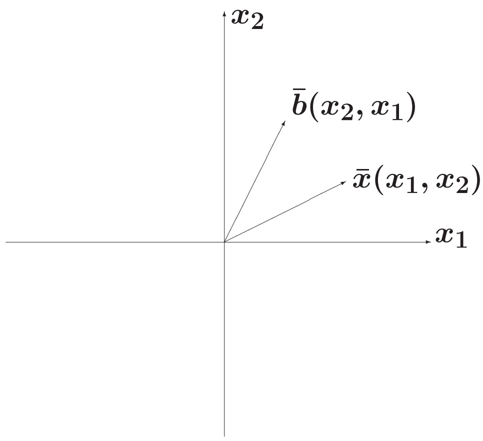
2. $A_2\bar{x} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -x_1 \\ x_2 \end{bmatrix}$



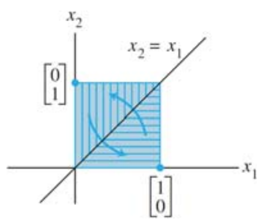
Reflection through the x_2-axis



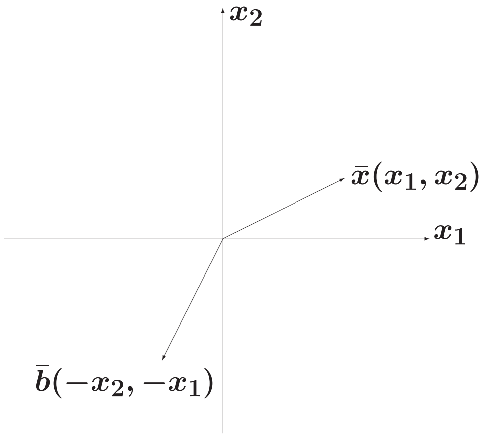
3. $A_3\bar{x} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_2 \\ x_1 \end{bmatrix}$



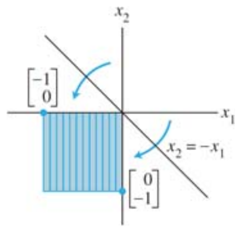
Reflection through the line $x_2 = x_1$



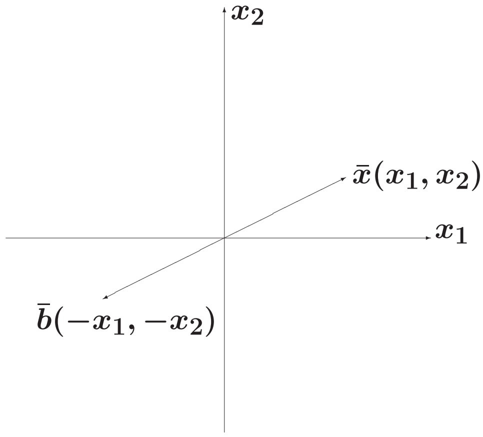
4. $A_4\bar{x} = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -x_2 \\ -x_1 \end{bmatrix}$



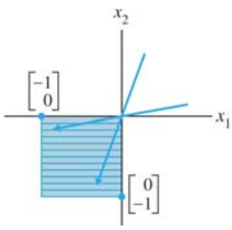
Reflection through the line $x_2 = -x_1$



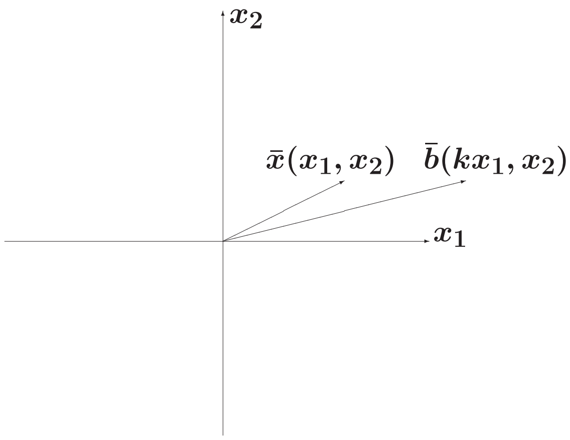
5. $A_5\bar{x} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -x_1 \\ -x_2 \end{bmatrix}$



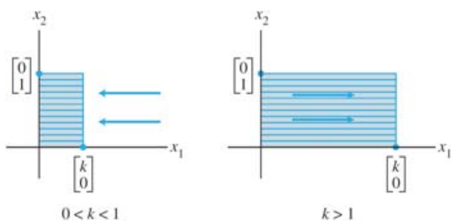
Reflection through the origin



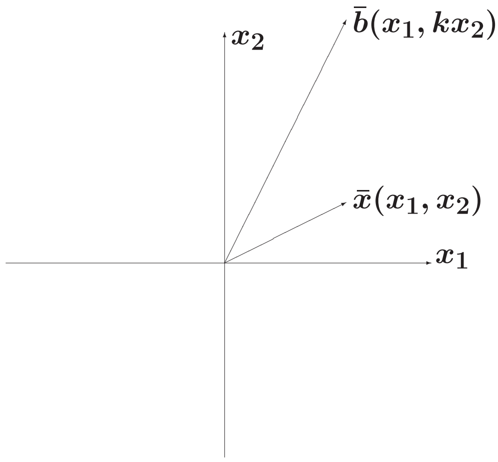
6. $B_1\bar{x} = \begin{bmatrix} k & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} kx_1 \\ x_2 \end{bmatrix}$



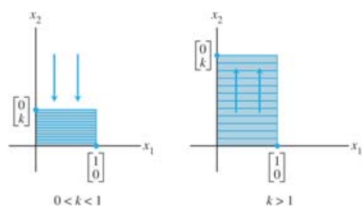
Horizontal expansion



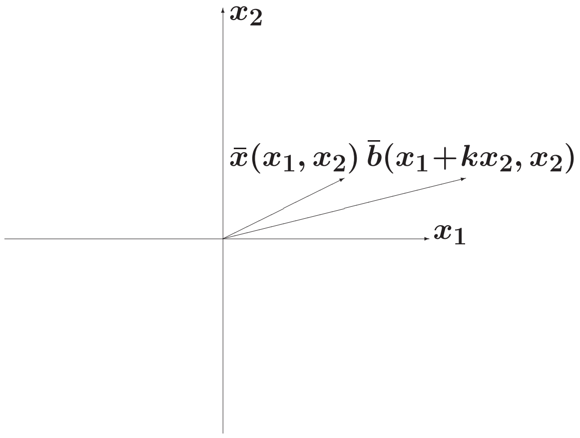
7. $B_2\bar{x} = \begin{bmatrix} 1 & 0 \\ 0 & k \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ kx_2 \end{bmatrix}$



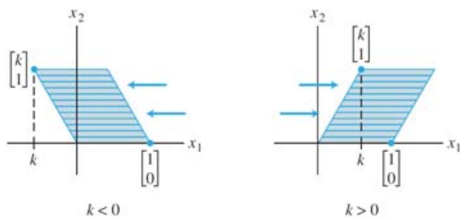
Vertical expansion



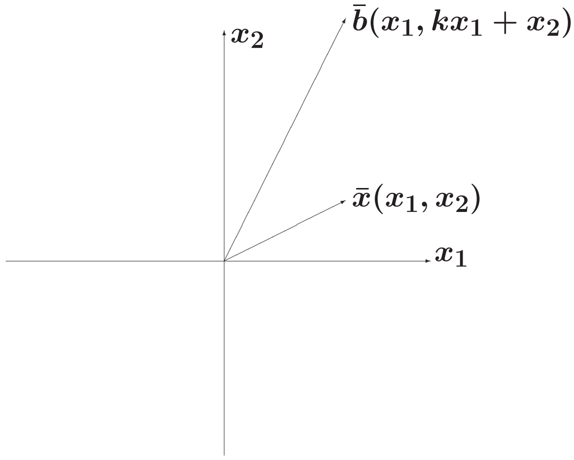
8. $C_1\bar{x} = \begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 + kx_2 \\ x_2 \end{bmatrix}$



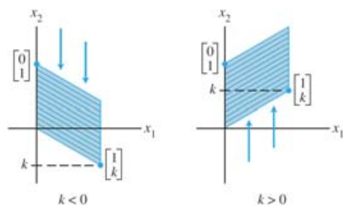
Horizontal shear



9. $C_2\bar{x} = \begin{bmatrix} 1 & 0 \\ k & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ kx_1 + x_2 \end{bmatrix}$



Vertical shear



LINEAR TRANSFORMATIONS

DEFINITION:

A transformation (or function, or mapping) T from R^n to R^m is a rule that assigns to each vector $\bar{x}$ from R^n a vector $T(\bar{x})$ in R^m. The set R^n is called the domain, and R^m is called the codomain of T.

EXAMPLE:

Let

$$A = \begin{bmatrix} 1 & -3 \\ 3 & 5 \\ -1 & 7 \end{bmatrix}, \quad \bar{u} = \begin{bmatrix} 2 \\ -1 \end{bmatrix},$$

$$T : R^2 \to R^3, \quad T(\bar{x}) = A\bar{x}.$$

Find $T(\bar{u})$.

SOLUTION:

We have:

$$T(\bar{u}) = \begin{bmatrix} 1 & -3 \\ 3 & 5 \\ -1 & 7 \end{bmatrix} \begin{bmatrix} 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 5 \\ 1 \\ -9 \end{bmatrix}.$$

DEFINITION:

A transformation T is linear if:

(i) $T(\bar{u}+\bar{v}) = T(\bar{u}) + T(\bar{v})$ for all $\bar{u}, \bar{v}$ in the domain of T

(ii) $T(c\bar{u}) = cT(\bar{u})$ for all $\bar{u}$ in the domain of T and all scalars c

THEOREM:

If T is a linear transformation, then

$$T(\bar{0}) = \bar{0}$$

and

$$T(c\bar{u} + d\bar{v}) = cT(\bar{u}) + dT(\bar{v})$$

for all vectors $\bar{u}, \bar{v}$ in the domain of T and all scalars c, d.

PROBLEM:

Find a matrix A such that for any $\bar{x}$ from R^2 we have

$$T(\bar{x}) = 3\bar{x}.$$

SOLUTION:

Let

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$

Then

$$T(\bar{x}) = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} ax_1 + bx_2 \\ cx_1 + dx_2 \end{bmatrix}$$

and

$$3\bar{x} = \begin{bmatrix} 3x_1 \\ 3x_2 \end{bmatrix}.$$

Therefore

$$\begin{bmatrix} ax_1 + bx_2 \\ cx_1 + dx_2 \end{bmatrix} = \begin{bmatrix} 3x_1 \\ 3x_2 \end{bmatrix}.$$

Hence,

$$a = 3, \; b = 0, \; c = 0, \; d = 3,$$

so

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}.$$