PROBLEM:
Let $\bar{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ and 

$A_1 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$, $A_2 = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$,

$A_3 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, $A_4 = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$,

$A_5 = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$

$B_1 = \begin{bmatrix} k & 0 \\ 0 & 1 \end{bmatrix}$, $B_2 = \begin{bmatrix} 1 & 0 \\ 0 & k \end{bmatrix}$

$C_1 = \begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix}$, $C_2 = \begin{bmatrix} 1 & 0 \\ k & 1 \end{bmatrix}$

Find $A_i\bar{x}$, $B_i\bar{x}$, $C_i\bar{x}$. Provide illustrations and geometric explanations.

1. $A_1\bar{x} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ -x_2 \end{bmatrix}$

2. $A_2\bar{x} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -x_1 \\ x_2 \end{bmatrix}$

3. $A_3\bar{x} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_2 \\ x_1 \end{bmatrix}$
4. \( A_4 \vec{x} = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -x_2 \\ -x_1 \end{bmatrix} \)

Reflection through the line \( x_2 = -x_1 \)

5. \( A_5 \vec{x} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -x_1 \\ -x_2 \end{bmatrix} \)

Reflection through the origin

6. \( B_1 \vec{x} = \begin{bmatrix} k & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} kx_1 \\ x_2 \end{bmatrix} \)

Horizontal expansion

7. \( B_2 \vec{x} = \begin{bmatrix} 1 & 0 \\ 0 & k \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ kx_2 \end{bmatrix} \)

Vertical expansion
8. \( C_1 \bar{x} = \begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 + kx_2 \\ x_2 \end{bmatrix} \)

Horizontal shear

9. \( C_2 \bar{x} = \begin{bmatrix} 1 & 0 \\ k & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ kx_1 + x_2 \end{bmatrix} \)

Vertical shear

**LINEAR TRANSFORMATIONS**

**DEFINITION:**

A transformation (or function, or mapping) \( T \) from \( \mathbb{R}^n \) to \( \mathbb{R}^m \) is a rule that assigns to each vector \( \bar{x} \) from \( \mathbb{R}^n \) a vector \( T(\bar{x}) \) in \( \mathbb{R}^m \). The set \( \mathbb{R}^n \) is called the domain, and \( \mathbb{R}^m \) is called the codomain of \( T \).

**EXAMPLE:**

Let

\[
A = \begin{bmatrix} 1 & -3 \\ 3 & 5 \\ -1 & 7 \end{bmatrix}, \quad \bar{u} = \begin{bmatrix} 2 \\ -1 \end{bmatrix},
\]

\( T : \mathbb{R}^2 \to \mathbb{R}^3, \quad T(\bar{x}) = A\bar{x}. \)

Find \( T(\bar{u}). \)

**SOLUTION:**

We have:

\[
T(\bar{u}) = \begin{bmatrix} 1 & -3 \\ 3 & 5 \\ -1 & 7 \end{bmatrix} \begin{bmatrix} 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 5 \\ 1 \\ -9 \end{bmatrix}.
\]
**DEFINITION:**

A transformation $T$ is **linear** if:

(i) $T(\bar{u} + \bar{v}) = T(\bar{u}) + T(\bar{v})$ for all $\bar{u}, \bar{v}$ in the domain of $T$

(ii) $T(c\bar{u}) = cT(\bar{u})$ for all $\bar{u}$ in the domain of $T$ and all scalars $c$

**THEOREM:**

If $T$ is a linear transformation, then

$T(\bar{0}) = \bar{0}$

and

$T(c\bar{u} + d\bar{v}) = cT(\bar{u}) +dT(\bar{v})$

for all vectors $\bar{u}, \bar{v}$ in the domain of $T$ and all scalars $c, d$.

**PROBLEM:**

Find a matrix $A$ such that for any $\bar{x}$ from $\mathbb{R}^2$ we have

$T(\bar{x}) = 3\bar{x}$.

**SOLUTION:**

Let

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$  

Then

$$T(\bar{x}) = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} ax_1 + bx_2 \\ cx_1 + dx_2 \end{bmatrix}$$

and

$$3\bar{x} = \begin{bmatrix} 3x_1 \\ 3x_2 \end{bmatrix}.$$  

Therefore

$$\begin{bmatrix} ax_1 + bx_2 \\ cx_1 + dx_2 \end{bmatrix} = \begin{bmatrix} 3x_1 \\ 3x_2 \end{bmatrix}.$$  

Hence,

$a = 3, \ b = 0, \ c = 0, \ d = 3$,

so

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}.$$