DEFINITION:

Vectors $\bar{v}_1, \ldots, \bar{v}_p$ are said to be linearly dependent if there exist scalars $c_1, \ldots, c_p$, not all zero, such that

$$c_1\bar{v}_1 + \ldots + c_p\bar{v}_p = \bar{0}.$$

Vectors $\bar{v}_1, \ldots, \bar{v}_p$ are said to be linearly independent if the vector equation

$$c_1\bar{v}_1 + \ldots + c_p\bar{v}_p = \bar{0}$$

has only the trivial solution.

EXAMPLE: Show that vectors

$$\bar{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}, \quad \bar{v}_2 = \begin{bmatrix} -2 \\ -1 \\ -4 \end{bmatrix}, \quad \bar{v}_3 = \begin{bmatrix} 3 \\ 5 \\ 6 \end{bmatrix}$$

are linearly dependent. Then show that vectors

$$\bar{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}, \quad \bar{v}_2 = \begin{bmatrix} -2 \\ -1 \\ -4 \end{bmatrix}, \quad \bar{v}_3 = \begin{bmatrix} 3 \\ 5 \\ 7 \end{bmatrix}$$

are linearly independent.

Solution: To show that the vectors

$$\bar{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}, \quad \bar{v}_2 = \begin{bmatrix} -2 \\ -1 \\ -4 \end{bmatrix}, \quad \bar{v}_3 = \begin{bmatrix} 3 \\ 5 \\ 6 \end{bmatrix}$$

are linearly dependent, we find c_1, c_2, c_3, not all zero, such that

$$c_1\bar{v}_1 + c_2\bar{v}_2 + c_3\bar{v}_3 = \bar{0}.$$

We have

$$\begin{bmatrix} 1 & -2 & 3 & 0 \\ 1 & -1 & 5 & 0 \\ 2 & -4 & 6 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 & 3 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 7 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

therefore

$$\begin{cases} c_1 + 7c_3 = 0 \\ c_2 + 2c_3 = 0 \\ c_3 \text{ is free} \end{cases} \implies \begin{cases} c_1 = -7c_3 \\ c_2 = -2c_3 \\ c_3 \text{ is free} \end{cases}$$

For example, if $c_3 = -1$, then $c_1 = 7$ and $c_2 = 2$, that is

$$7\bar{v}_1 + 2\bar{v}_2 - \bar{v}_3 = \bar{0}$$

We now show that the vectors

$$\bar{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}, \quad \bar{v}_2 = \begin{bmatrix} -2 \\ -1 \\ -4 \end{bmatrix}, \quad \bar{v}_3 = \begin{bmatrix} 3 \\ 5 \\ 7 \end{bmatrix}$$

are linearly independent. We have

$$\begin{bmatrix} 1 & -2 & 3 & 0 \\ 1 & -1 & 5 & 0 \\ 2 & -4 & 7 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 & 3 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

Since there are no free variables, the vector equation

$$c_1\bar{v}_1 + c_2\bar{v}_2 + c_3\bar{v}_3 = \bar{0}$$

has only the trivial solution.

THEOREM:

Let A be an $m \times n$ matrix. Then the following statements are logically equivalent:

1. A homogeneous system $A\bar{x} = \bar{0}$ has only the trivial solution.
2. There are no free variables.
3. Number of columns of A = Number of pivot positions.
4. The columns of a matrix A are linearly independent.

THEOREM:

Let A be an $m \times n$ matrix. Then the following statements are logically equivalent:

1. A homogeneous system $A\bar{x} = \bar{0}$ has a nontrivial solution.
2. There are free variables.
3. Number of columns of $A >$ Number of pivot positions.
4. The columns of a matrix A are linearly dependent.
5. At least one column of A is a linear combination of other columns.

1. Let
$$\bar{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad \bar{v}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad \bar{v}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$
(a) Are $\bar{v}_1, \bar{v}_2, \bar{v}_3$ linearly independent?
(b) Does $\{\bar{v}_1, \bar{v}_2, \bar{v}_3\}$ span R^3 ?

2. Let
$$\bar{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad \bar{v}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad \bar{v}_3 = \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix}.$$
(a) Are $\bar{v}_1, \bar{v}_2, \bar{v}_3$ linearly independent?
(b) Does $\{\bar{v}_1, \bar{v}_2, \bar{v}_3\}$ span R^3 ?

3. Let
$$\bar{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \bar{v}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \bar{v}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \bar{v}_4 = \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix}.$$
(a) Are $\bar{v}_1, \bar{v}_2, \bar{v}_3, \bar{v}_4$ linearly independent?
(b) Does $\{\bar{v}_1, \bar{v}_2, \bar{v}_3, \bar{v}_4\}$ span R^3 ?

4. Let
$$\bar{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \bar{v}_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$
(a) Are $\bar{v}_1, \bar{v}_2$ linearly independent?
(b) Does $\{\bar{v}_1, \bar{v}_2\}$ span R^3 ?

1. Let
$$\bar{v}_1 = \begin{bmatrix} 5 \\ 0 \\ 0 \end{bmatrix}, \quad \bar{v}_2 = \begin{bmatrix} 7 \\ 2 \\ -6 \end{bmatrix}, \quad \bar{v}_3 = \begin{bmatrix} 9 \\ 4 \\ -8 \end{bmatrix}.$$
(a) Are $\bar{v}_1, \bar{v}_2, \bar{v}_3$ linearly independent?
(b) Does $\{\bar{v}_1, \bar{v}_2, \bar{v}_3\}$ span R^3 ?

2. Let
$$\bar{v}_1 = \begin{bmatrix} 5 \\ 0 \\ 0 \end{bmatrix}, \quad \bar{v}_2 = \begin{bmatrix} 7 \\ 2 \\ -6 \end{bmatrix}, \quad \bar{v}_3 = \begin{bmatrix} 9 \\ 4 \\ -12 \end{bmatrix}.$$
(a) Are $\bar{v}_1, \bar{v}_2, \bar{v}_3$ linearly independent?
(b) Does $\{\bar{v}_1, \bar{v}_2, \bar{v}_3\}$ span R^3 ?

3. Let
$$\bar{v}_1 = \begin{bmatrix} 5 \\ 0 \\ 0 \end{bmatrix}, \bar{v}_2 = \begin{bmatrix} 7 \\ 2 \\ -6 \end{bmatrix}, \bar{v}_3 = \begin{bmatrix} 9 \\ 4 \\ -8 \end{bmatrix}, \bar{v}_4 = \begin{bmatrix} 9 \\ 4 \\ -12 \end{bmatrix}.$$
(a) Are $\bar{v}_1, \bar{v}_2, \bar{v}_3, \bar{v}_4$ linearly independent?
(b) Does $\{\bar{v}_1, \bar{v}_2, \bar{v}_3, \bar{v}_4\}$ span R^3 ?

4. Let
$$\bar{v}_1 = \begin{bmatrix} 5 \\ 0 \\ 0 \end{bmatrix}, \bar{v}_2 = \begin{bmatrix} 7 \\ 2 \\ -6 \end{bmatrix}.$$
(a) Are $\bar{v}_1, \bar{v}_2$ linearly independent?
(b) Does $\{\bar{v}_1, \bar{v}_2\}$ span R^3 ?