$$\begin{cases} x_1 + 3x_2 + 4x_3 = 7 \\ 3x_1 + 9x_2 + 7x_3 = 6 \end{cases}$$

$$\Downarrow$$

$$\begin{bmatrix} 1 & 3 & 4 & 7 \\ 3 & 9 & 7 & 6 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 & 4 & 7 \\ 0 & 0 & -5 & -15 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 3 & 4 & 7 \\ 0 & 0 & 1 & 3 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 & 0 & -5 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

$$\Downarrow$$

$$\begin{cases} x_1 + 3x_2 \qquad = -5 \\ \qquad\qquad x_3 = 3 \end{cases}$$

$$\Downarrow$$

$$\begin{cases} x_1 = -5 - 3x_2 \\ x_3 = 3 \\ x_2 \text{ is free} \end{cases}$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -5 - 3x_2 \\ x_2 \\ 3 \end{bmatrix} = \begin{bmatrix} -5 - 3 \cdot x_2 \\ 0 + 1 \cdot x_2 \\ 3 + 0 \cdot x_2 \end{bmatrix}$$

$$= \begin{bmatrix} -5 \\ 0 \\ 3 \end{bmatrix} + x_2 \begin{bmatrix} -3 \\ 1 \\ 0 \end{bmatrix}$$

DEFINITION:

A system of linear equations is said to be homogeneous if it can be written in the form $A\bar{x} = \bar{0}$. Otherwise, it is non-homogeneous.

EXAMPLE:

$$\begin{cases} 3x_1 + 5x_2 = 0 \\ 6x_1 + 2x_2 = 0 \end{cases} \qquad \text{HOMOGENEOUS}$$

$$\begin{cases} 3x_1 + 5x_2 = 1 \\ 6x_1 + 2x_2 = 0 \end{cases} \qquad \text{NONHOMOGEN.}$$

THEOREM:

Suppose the equation $A\bar{x} = \bar{b}$ is consistent for some given $\bar{b}$, and let $\bar{p}$ be a solution. Then the solution set of $A\bar{x} = \bar{b}$ is the set of all vectors of the form

$$\bar{w} = \bar{p} + \bar{v}_h,$$

where $\bar{v}_h$ is any solution of the homogeneous equation $A\bar{x} = \bar{0}$.

THEOREM:

Let A be an $m \times n$ matrix. Then the following statements are logically equivalent:

1. A homogeneous system $A\bar{x} = \bar{0}$ has only the trivial solution.

2. There are no free variables.

3. Number of columns of A = Number of pivot positions.

PROBLEM:

Let

$$A = \begin{bmatrix} 3 & 5 & -4 \\ -3 & -2 & 4 \\ 6 & 1 & -8 \end{bmatrix} \quad \text{and} \quad \bar{b} = \begin{bmatrix} 7 \\ -1 \\ -4 \end{bmatrix}$$

1. Is $\bar{b}$ in the subset of R^3 spanned by the columns of A?
2. Do columns of A span R^3?
3. Describe all solutions of $A\bar{x} = \bar{b}$.

SOLUTION:

$$\begin{bmatrix} 3 & 5 & -4 & 7 \\ -3 & -2 & 4 & -1 \\ 6 & 1 & -8 & -4 \end{bmatrix} \sim \begin{bmatrix} 3 & 5 & -4 & 7 \\ 0 & 3 & 0 & 6 \\ 0 & -9 & 0 & -18 \end{bmatrix}$$

$$\sim \begin{bmatrix} 3 & 5 & -4 & 7 \\ 0 & 3 & 0 & 6 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & \frac{5}{3} & -\frac{4}{3} & \frac{7}{3} \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 0 & -\frac{4}{3} & -1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

1. Is $\bar{b}$ in the subset of R^3 spanned by the columns of A? Yes, since the system $A\bar{x} = \bar{b}$ is consistent.
2. Do columns of A span R^3? No, since the number of pivots (two) is less than three.
3. Describe all solutions of $A\bar{x} = \bar{b}$.

$$\begin{cases} x_1 - \frac{4}{3}x_3 = -1 \\ x_2 = 2 \end{cases} \Rightarrow \begin{cases} x_1 = -1 + \frac{4}{3}x_3 \\ x_2 = 2 \\ x_3 \text{ is free} \end{cases}$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -1 + \frac{4}{3}x_3 \\ 2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -1 + \frac{4}{3} \cdot x_3 \\ 2 + 0 \cdot x_3 \\ 0 + 1 \cdot x_3 \end{bmatrix}$$

$$= \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} \frac{4}{3} \\ 0 \\ 1 \end{bmatrix}$$