**DEFINITION**:

A matrix with only one column is called a column vector, or simply a vector.

**DEFINITION**:

The vector $\bar{y}$ defined by

$$\bar{y} = c_1\bar{v_1} + \ldots + c_p\bar{v_p},$$

where $\bar{v_1}, \ldots, \bar{v_p}$ are vectors and $c_1, \ldots, c_p$ are scalars, is called a linear combination of vectors $\bar{v_1}, \ldots, \bar{v_p}$.

**DEFINITION**:

The set of all combinations of $\bar{v_1}, \ldots, \bar{v_p}$ is denoted by Span$\{\bar{v_1}, \ldots, \bar{v_p}\}$ and is called the subset of $R^n$ spanned (or generated) by $\bar{v_1}, \ldots, \bar{v_p}$.

$$\begin{cases} x_1 - 2x_2 + 3x_3 + 17x_4 = 18 \\ 2x_1 - 4x_2 + 9x_3 + 46x_4 = 57 \\ 3x_1 - 6x_2 + 4x_3 + 31x_4 = 19 \end{cases}$$

$$x_1\begin{bmatrix}1\\2\\3\end{bmatrix} + x_2\begin{bmatrix}-2\\-4\\-6\end{bmatrix} + x_3\begin{bmatrix}3\\9\\4\end{bmatrix} + x_4\begin{bmatrix}17\\46\\31\end{bmatrix} = \begin{bmatrix}18\\57\\19\end{bmatrix}$$

$$\begin{bmatrix}1 & -2 & 3 & 17\\2 & -4 & 9 & 46\\3 & -6 & 4 & 31\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\\x_4\end{bmatrix} = \begin{bmatrix}x_1 - 2x_2 + 3x_3 + 17x_4\\2x_1 - 4x_2 + 9x_3 + 46x_4\\3x_1 - 6x_2 + 4x_3 + 31x_4\end{bmatrix} = \begin{bmatrix}18\\57\\19\end{bmatrix}$$

Row-Vector Rule for Computing $A\bar{x}$:

If a product $A\bar{x}$ is defined, then the $i$th entry in $A\bar{x}$ is the sum of the products of corresponding entries from row $i$ of $A$ and from the vector $\bar{x}$.

**EXAMPLE**:

1. $$\begin{bmatrix}1 & 2 & -1\\0 & -5 & 3\end{bmatrix}\begin{bmatrix}4\\3\\7\end{bmatrix} = \begin{bmatrix}1\cdot 4 + 2\cdot 3 + (-1)\cdot 7\\0\cdot 4 + (-5)\cdot 3 + 3\cdot 7\end{bmatrix} = \begin{bmatrix}3\\6\end{bmatrix}$$

2. $$\begin{bmatrix}2 & -3 & -9 & 11\\-3 & 0 & 6 & -4\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\\x_4\end{bmatrix} = \begin{bmatrix}2x_1 - 3x_2 - 9x_3 + 11x_4\\-3x_1 + 6x_3 - 4x_4\end{bmatrix}$$

3. $$\begin{bmatrix}2 & -3 & -9 & 11\\-3 & 0 & 6 & -4\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix} = ???$$

**THEOREM**:

Let $A$ be an $m \times n$ matrix. Then the following statements are logically equivalent:

1. For each $\bar{b}$ in $R^m$, the equation $A\bar{x} = \bar{b}$ has a solution.

2. Each $\bar{b}$ in $R^m$ is a linear combination of the columns of $A$.

3. The columns of $A$ span $R^m$.

4. $A$ has a pivot position in every row, i.e. $A$ has $m$ pivot positions.

$$x_1 \begin{bmatrix} 1 \\ 3 \end{bmatrix} + x_2 \begin{bmatrix} 3 \\ 9 \end{bmatrix} + x_3 \begin{bmatrix} 4 \\ 7 \end{bmatrix} = \begin{bmatrix} 7 \\ 6 \end{bmatrix}$$

Let

$$\bar{v}_1 = \begin{bmatrix} 1 \\ 3 \end{bmatrix}, \quad \bar{v}_2 = \begin{bmatrix} 3 \\ 9 \end{bmatrix}, \quad \bar{v}_3 = \begin{bmatrix} 4 \\ 7 \end{bmatrix}.$$

(a) Does $\{\bar{v}_1, \bar{v}_2, \bar{v}_3\}$ span $R^2$ ?
(b) Does $\{\bar{v}_1, \bar{v}_2, \bar{v}_3\}$ span $R^3$ ?
(c) Is $\begin{bmatrix} 7 \\ 6 \end{bmatrix}$ in Span $(\{\bar{v}_1, \bar{v}_2, \bar{v}_3\})$?
(d) Is $\begin{bmatrix} 7 \\ 32 \end{bmatrix}$ in Span $(\{\bar{v}_1, \bar{v}_2, \bar{v}_3\})$?
(e) Does

$$x_1\bar{v}_1 + x_2\bar{v}_2 + x_3\bar{v}_3 = \begin{bmatrix} 7 \\ 6 \end{bmatrix}$$

have a solution?
(f) Does

$$x_1\bar{v}_1 + x_2\bar{v}_2 + x_3\bar{v}_3 = \begin{bmatrix} 7 \\ 32 \end{bmatrix}$$

have a solution?

$$\begin{cases} x_1 + 3x_2 + 4x_3 = 7 \\ 3x_1 + 9x_2 + 7x_3 = 6 \end{cases}$$

$$\Downarrow$$

$$\begin{bmatrix} 1 & 3 & 4 & 7 \\ 3 & 9 & 7 & 6 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 & 4 & 7 \\ 0 & 0 & -5 & -15 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 3 & 4 & 7 \\ 0 & 0 & 1 & 3 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 & 0 & -5 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

$$\Downarrow$$

$$\begin{cases} x_1 + 3x_2 \qquad = -5 \\ \qquad\qquad x_3 = 3 \end{cases}$$

$$\Downarrow$$

$$\begin{cases} x_1 = -5 - 3x_2 \\ x_3 = 3 \\ x_2 \text{ is free} \end{cases}$$

**PROBLEMS**:

1.

$$\bar{v}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad \bar{v}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

(a) Does $\{\bar{v}_1, \bar{v}_2\}$ span $R^2$ ?
(b) Does $\{\bar{v}_1, \bar{v}_2\}$ span $R^3$ ?
(c) Is $\begin{bmatrix} 3 \\ 2 \end{bmatrix}$ in Span $(\{\bar{v}_1, \bar{v}_2\})$?
(d) Does

$$x_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$$

have a solution?

**2.**

$$\bar{v}_1 = \begin{bmatrix} 1 \\ 4 \end{bmatrix}, \quad \bar{v}_2 = \begin{bmatrix} 2 \\ 7 \end{bmatrix}.$$

(a) Does $\{\bar{v}_1, \bar{v}_2\}$ span $R^2$ ?

(b) Does $\{\bar{v}_1, \bar{v}_2\}$ span $R^3$ ?

(c) Is $\begin{bmatrix} 3 \\ 1 \end{bmatrix}$ in Span $(\{\bar{v}_1, \bar{v}_2\})$?

(d) Does

$$x_1 \begin{bmatrix} 1 \\ 4 \end{bmatrix} + x_2 \begin{bmatrix} 2 \\ 7 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

have a solution?

(e) Is $\begin{bmatrix} 100 \\ 165889 \end{bmatrix}$ in Span $(\{\bar{v}_1, \bar{v}_2\})$?

(f) Does

$$x_1 \begin{bmatrix} 1 \\ 4 \end{bmatrix} + x_2 \begin{bmatrix} 2 \\ 7 \end{bmatrix} = \begin{bmatrix} 100 \\ 165889 \end{bmatrix}$$

have a solution?

**3.**

$$\bar{v}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \quad \bar{v}_2 = \begin{bmatrix} 5 \\ 10 \end{bmatrix}, \quad \bar{v}_3 = \begin{bmatrix} -3 \\ -6 \end{bmatrix}.$$

(a) Does $\{\bar{v}_1, \bar{v}_2, \bar{v}_3\}$ span $R^2$ ?

(b) Does $\{\bar{v}_1, \bar{v}_2, \bar{v}_3\}$ span $R^3$ ?

(c) Is $\begin{bmatrix} -4 \\ -8 \end{bmatrix}$ in Span $(\{\bar{v}_1, \bar{v}_2, \bar{v}_3\})$?

(d) Is $\begin{bmatrix} 5 \\ 7 \end{bmatrix}$ in Span $(\{\bar{v}_1, \bar{v}_2, \bar{v}_3\})$?

(e) Does

$$x_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + x_2 \begin{bmatrix} 5 \\ 10 \end{bmatrix} + x_3 \begin{bmatrix} -3 \\ -6 \end{bmatrix} = \begin{bmatrix} -4 \\ -8 \end{bmatrix}$$

have a solution?

(f) Does

$$x_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + x_2 \begin{bmatrix} 5 \\ 10 \end{bmatrix} + x_3 \begin{bmatrix} -3 \\ -6 \end{bmatrix} = \begin{bmatrix} 5 \\ 7 \end{bmatrix}$$

have a solution?