ECHELON FORM:

$$\begin{bmatrix} \blacksquare & * & * & * & * & * & * & * & * & * \\ 0 & 0 & \blacksquare & * & * & * & * & * & * & * \\ 0 & 0 & 0 & \blacksquare & * & * & * & * & * & * \\ 0 & 0 & 0 & 0 & \blacksquare & * & * & * & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \blacksquare & * \end{bmatrix}$$

1. All nonzero rows are above any rows of all zeros.
2. Each leading entry of a row is in a column to the right of the leading entry of the row above it.
3. All entries in a column below a leading entry are zeros.

REDUCED ECHELON FORM:

$$\begin{bmatrix} 1 & * & 0 & 0 & 0 & * & * & * & 0 & * \\ 0 & 0 & 1 & 0 & 0 & * & * & * & 0 & * \\ 0 & 0 & 0 & 1 & 0 & * & * & * & 0 & * \\ 0 & 0 & 0 & 0 & 1 & * & * & * & 0 & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & * \end{bmatrix}$$

4. The leading entry in each nonzero row is 1.
5. Each leading 1 is the only nonzero entry in its column.

DEFINITION:

A pivot position in a matrix is a location in A that corresponds to a leading 1 in the reduced echelon form of A.
A pivot column is a column of A that contains a pivot position.

DEFINITION:

The variables corresponding to pivot columns are called basic variables.
The other variables are called free variables.

ELEMENTARY ROW OPERATIONS:

1. Replace one row by the sum of itself and a multiple of another row.
2. Interchange two rows.
3. Multiply all entries in a row by a nonzero constant.

$$\begin{cases} x_1 - 2x_2 + x_3 = 0 \\ 2x_2 - 8x_3 = 8 \\ 4x_1 - 5x_2 - 9x_3 = 9 \end{cases} \Rightarrow \begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ 4 & -5 & -9 & 9 \end{bmatrix}$$

$$\Downarrow$$

$$\begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ 0 & 3 & -13 & 9 \end{bmatrix}$$

$$\Downarrow$$

$$\begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 1 & -4 & 4 \\ 0 & 3 & -13 & 9 \end{bmatrix}$$

$$\Downarrow$$

$$\begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 1 & -4 & 4 \\ 0 & 0 & -1 & -3 \end{bmatrix}$$

$$\Downarrow$$

$$\begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 1 & -4 & 4 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

$$\Downarrow$$

$$\begin{bmatrix} 1 & -2 & 0 & -3 \\ 0 & 1 & 0 & 16 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

$$\Downarrow$$

$$\begin{cases} x_1 = 29 \\ x_2 = 16 \\ x_3 = 3 \end{cases} \Leftarrow \begin{bmatrix} 1 & 0 & 0 & 29 \\ 0 & 1 & 0 & 16 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

$$\begin{cases} 2x_1 + 3x_2 + 8x_4 = 0 \\ x_2 - x_3 + 3x_4 = 0 \\ x_3 + 2x_4 = 1 \\ x_1 + x_4 = -24 \end{cases}$$

$$\Downarrow$$

$$\begin{bmatrix} 2 & 3 & 0 & 8 & 0 \\ 0 & 1 & -1 & 3 & 0 \\ 0 & 0 & 1 & 2 & 1 \\ 1 & 0 & 0 & 1 & -24 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 1 & -24 \\ 2 & 3 & 0 & 8 & 0 \\ 0 & 1 & -1 & 3 & 0 \\ 0 & 0 & 1 & 2 & 1 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 1 & -24 \\ 0 & 3 & 0 & 6 & 48 \\ 0 & 1 & -1 & 3 & 0 \\ 0 & 0 & 1 & 2 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 1 & -24 \\ 0 & 1 & 0 & 2 & 16 \\ 0 & 1 & -1 & 3 & 0 \\ 0 & 0 & 1 & 2 & 1 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 1 & -24 \\ 0 & 1 & 0 & 2 & 16 \\ 0 & 0 & -1 & 1 & -16 \\ 0 & 0 & 1 & 2 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 1 & -24 \\ 0 & 1 & 0 & 2 & 16 \\ 0 & 0 & 1 & -1 & 16 \\ 0 & 0 & 1 & 2 & 1 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 1 & -24 \\ 0 & 1 & 0 & 2 & 16 \\ 0 & 0 & 1 & -1 & 16 \\ 0 & 0 & 0 & 3 & -15 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 1 & -24 \\ 0 & 1 & 0 & 2 & 16 \\ 0 & 0 & 1 & -1 & 16 \\ 0 & 0 & 0 & 1 & -5 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 & -19 \\ 0 & 1 & 0 & 0 & 26 \\ 0 & 0 & 1 & 0 & 11 \\ 0 & 0 & 0 & 1 & -5 \end{bmatrix} \Rightarrow \begin{cases} x_1 = -19 \\ x_2 = 26 \\ x_3 = 11 \\ x_4 = -5 \end{cases}$$

$$\begin{cases} 2x_1 - x_2 = -1 \\ x_1 + 2x_2 - x_3 = -2 \\ x_2 + x_3 = -2 \end{cases}$$

$$\Downarrow$$

$$\begin{bmatrix} 2 & -1 & 0 & -1 \\ 1 & 2 & -1 & -2 \\ 0 & 1 & 1 & -2 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & -1 & -2 \\ 2 & -1 & 0 & -1 \\ 0 & 1 & 1 & -2 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 2 & -1 & -2 \\ 0 & -5 & 2 & 3 \\ 0 & 1 & 1 & -2 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & -1 & -2 \\ 0 & 1 & 1 & -2 \\ 0 & -5 & 2 & 3 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 2 & -1 & -2 \\ 0 & 1 & 1 & -2 \\ 0 & 0 & 7 & -7 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & -1 & -2 \\ 0 & 1 & 1 & -2 \\ 0 & 0 & 1 & -1 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 2 & 0 & -3 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \end{bmatrix}$$

$$\Downarrow$$

$$\begin{cases} x_1 = -1 \\ x_2 = -1 \\ x_3 = -1 \end{cases}$$

$$\begin{cases} 2x_1 + 5x_2 - 8x_3 = 8 \\ 4x_1 + 3x_2 - 9x_3 = 9 \\ 2x_1 + 3x_2 - 5x_3 = 7 \\ x_1 + 8x_2 - 7x_3 = 12 \end{cases}$$

$$\Downarrow$$

$$\begin{bmatrix} 2 & 5 & -8 & 8 \\ 4 & 3 & -9 & 9 \\ 2 & 3 & -5 & 7 \\ 1 & 8 & -7 & 12 \end{bmatrix} \sim \begin{bmatrix} 1 & 8 & -7 & 12 \\ 4 & 3 & -9 & 9 \\ 2 & 3 & -5 & 7 \\ 2 & 5 & -8 & 8 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 8 & -7 & 12 \\ 0 & -7 & 7 & -7 \\ 0 & -2 & 3 & -1 \\ 2 & 5 & -8 & 8 \end{bmatrix} \sim \begin{bmatrix} 1 & 8 & -7 & 12 \\ 0 & -7 & 7 & -7 \\ 0 & -2 & 3 & -1 \\ 0 & -11 & 6 & -16 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 8 & -7 & 12 \\ 0 & 1 & -1 & 1 \\ 0 & 2 & -3 & 1 \\ 0 & 11 & -6 & 16 \end{bmatrix} \sim \begin{bmatrix} 1 & 8 & -7 & 12 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & -1 & -1 \\ 0 & 0 & 5 & 5 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 8 & -7 & 12 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 8 & 0 & 19 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \Rightarrow \begin{cases} x_1 = 3 \\ x_2 = 2 \\ x_3 = 1 \end{cases}$$