Representations of Integers

THEOREM: Let b be a positive integer with $b > 1$. Then every positive integer n can be written uniquely in the form

$$n = a_k b^k + a_{k-1} b^{k-1} + \ldots + a_1 b + a_0$$

where k is a nonnegative integer, a_j is an integer with $0 \leq a_j \leq b-1$ for $j = 0, 1, \ldots, k$, and the initial coefficient $a_k \neq 0$.

REMARK: The Theorem above can be extended to the case where n is a negative integer and (or) b is a negative integer with $b < -1$.

COROLLARY: Every positive integer may be represented as the sum of distinct powers of 2.

NOTATION: To distinguish representations of integers with different bases, we use a special notation. We write

$$(a_k a_{k-1} \ldots a_1 a_0)_b$$

to represent the number

$$a_k b^k + a_{k-1} b^{k-1} + \ldots + a_1 b + a_0$$

EXAMPLE: To illustrate base b notation, note that

$$(236)_7 = 2 \cdot 7^2 + 3 \cdot 7 + 6 = 125, \qquad (10010011)_2 = 1 \cdot 2^7 + 1 \cdot 2^4 + 1 \cdot 2 + 1 = 147$$

EXAMPLE: To find the base 2 expansion of 1864, base -2 expansion of 1864 and base -2 expansion of -1864, we use the Division Algorithm successively:

$$\begin{aligned}
1864 &= 2 \cdot 932 + 0 \\
932 &= 2 \cdot 466 + 0 \\
466 &= 2 \cdot 233 + 0 \\
233 &= 2 \cdot 116 + 1 \\
116 &= 2 \cdot 58 + 0 \\
58 &= 2 \cdot 29 + 0 \\
29 &= 2 \cdot 14 + 1 \\
14 &= 2 \cdot 7 + 0 \\
7 &= 2 \cdot 3 + 1 \\
3 &= 2 \cdot 1 + 1 \\
1 &= 2 \cdot 0 + 1
\end{aligned}$$

$$\begin{aligned}
1864 &= (-2) \cdot (-932) + 0 \\
-932 &= (-2) \cdot 466 + 0 \\
466 &= (-2) \cdot (-233) + 0 \\
-233 &= (-2) \cdot 117 + 1 \\
117 &= (-2) \cdot (-58) + 1 \\
-58 &= (-2) \cdot 29 + 0 \\
29 &= (-2) \cdot (-14) + 1 \\
-14 &= (-2) \cdot 7 + 0 \\
7 &= (-2) \cdot (-3) + 1 \\
-3 &= (-2) \cdot 2 + 1 \\
2 &= (-2) \cdot (-1) + 0 \\
-1 &= (-2) \cdot 1 + 1 \\
1 &= (-2) \cdot 0 + 1
\end{aligned}$$

$$\begin{aligned}
-1864 &= (-2) \cdot 932 + 0 \\
932 &= (-2) \cdot (-466) + 0 \\
-466 &= (-2) \cdot 233 + 0 \\
233 &= (-2) \cdot (-116) + 1 \\
-116 &= (-2) \cdot 58 + 0 \\
58 &= (-2) \cdot (-29) + 0 \\
-29 &= (-2) \cdot 15 + 1 \\
15 &= (-2) \cdot (-7) + 1 \\
-7 &= (-2) \cdot 4 + 1 \\
4 &= (-2) \cdot (-2) + 0 \\
-2 &= (-2) \cdot 1 + 0 \\
1 &= (-2) \cdot 0 + 1
\end{aligned}$$

Reading *up* the remainders of these divisions, we get

$$(1864)_{10} = (11101001000)_2 = (1101101011000)_{-2} \qquad \text{and} \qquad (-1864)_{10} = (100111001000)_{-2}$$

Check:

$$1864 = 1 \cdot 2^{10} + 1 \cdot 2^9 + 1 \cdot 2^8 + 1 \cdot 2^6 + 1 \cdot 2^3$$

$$1864 = 1 \cdot (-2)^{12} + 1 \cdot (-2)^{11} + 1 \cdot (-2)^9 + 1 \cdot (-2)^8 + 1 \cdot (-2)^6 + 1 \cdot (-2)^4 + 1 \cdot (-2)^3$$

$$-1864 = 1 \cdot (-2)^{11} + 1 \cdot (-2)^8 + 1 \cdot (-2)^7 + 1 \cdot (-2)^6 + 1 \cdot (-2)^3$$

IMPORTANT: To find the base 2 expansion of -1864, we do *not* use the Division Algorithm. Instead, we simply put minus in front of 11101001000:

$$(-1864)_{10} = (-11101001000)_2$$

Computers use base 8 or base 16 for display purposes. In base 16 (hexadecimal) notation there are l6 digits, usually denoted by

$$0,\ 1,\ 2,\ 3,\ 4,\ 5,\ 6,\ 7,\ 8,\ 9,\ A,\ B,\ C,\ D,\ E,\ F$$

The letters A, B, C, D, E, and F are used to represent the digits that correspond to $10, 11, 12, 13, 14$, and 15 (written in decimal notation).

EXAMPLE: To convert $(A35B0F)_{16}$ from hexadecimal to decimal notation, we write

$$(A35B0F)_{16} = 10 \cdot 16^5 + 3 \cdot 16^4 + 5 \cdot 16^3 + 11 \cdot 16^2 + 0 \cdot 16 + 15 = (10705679)_{10}$$

A simple conversion is possible between binary and hexadecimal notation. We can write each hex digit as a block of four binary digits according to the correspondence given in the Table below:

Hex Digit	Binary Digits	Hex Digit	Binary Digits
0	0000	8	1000
1	0001	9	1001
2	0010	A	1010
3	0011	B	1011
4	0100	C	1100
5	0101	D	1101
6	0110	E	1110
7	0111	F	1111

EXAMPLE: An example of conversion from hex to binary is

$$(2FB3)_{16} = (10111110110011)_2$$

Each hex digit is converted to a block of four binary digits (the initial zeros in the initial block $(0010)_2$ corresponding to the digit $(2)_{16}$ are omitted).

To convert from binary to hex, consider $(11110111101001)_2$. We break this into blocks of four starting from the right. The blocks are, from right to left, 1001, 1110, 1101, and 0011 (we add the initial zeros). Translating each block to hex, we obtain $(3DE9)_{16}$.

We note that a conversion between two different bases is as easy as binary hex conversion, whenever one of the bases is a power of the other.