The Fibonacci Numbers

DEFINITION: The **Fibonacci sequence** is defined recursively by $f_1 = 1, f_2 = 1$, and

$$f_n = f_{n-1} + f_{n-2} \quad \text{for} \quad n \geq 3$$

The terms of this sequence are called the **Fibonacci numbers**.

The Fibonacci s equence begins with the integers

$$1, \quad 1, \quad 2, \quad 3, \quad 5, \quad 8, \quad 13, \quad 21, \quad 34, \quad 55, \quad 89, \quad 144, \ldots$$

REMARK: We can define the value $f_0 = 0$ so that $f_2 = f_1 + f_0$.

EXAMPLE: For any positive integer n,

$$\sum_{k=1}^{n} f_k = f_{n+2} - 1$$

Proof: Since

$$f_{k+2} = f_{k+1} + f_k$$

it follows that

$$f_k = f_{k+2} - f_{k+1}$$

therefore

$$\begin{aligned}
\sum_{k=1}^{n} f_k &= \sum_{k=1}^{n} (f_{k+2} - f_{k+1}) \\
&= (f_3 - f_2) + (f_4 - f_3) + (f_5 - f_4) + \ldots + (f_{n+1} - f_n) + (f_{n+2} - f_{n+1}) \\
&= f_{n+2} - f_2 \\
&= f_{n+2} - 1
\end{aligned}$$

EXAMPLE: For any positive integer n,

$$f_{n+3} + f_n = 2f_{n+2} \tag{1}$$

Proof 1: We have

$$f_{n+3} + f_n = f_{n+2} + f_{n+1} + f_n = f_{n+2} + f_{n+2} = 2f_{n+2}$$

Proof 2: We have

$$f_n + f_{n+1} = f_{n+2} \tag{2}$$

and

$$f_{n+1} + f_{n+2} = f_{n+3} \tag{3}$$

Subtracting (3) from (2) we get

$$f_n - f_{n+2} = f_{n+2} - f_{n+3}$$

which can be rewritten as (1).

EXAMPLE: For any positive integer n,

$$f_{2n} = f_{n+1}^2 - f_{n-1}^2 \tag{4}$$

Proof:

STEP 1: For $n = 1$ (4) becomes

$$f_2 = f_2^2 - f_0^2$$

which is true, since $1 = 1^2 - 0^2$. Similarly, for $n = 2$ (4) becomes

$$f_4 = f_3^2 - f_1^2$$

which is true, since $3 = 2^2 - 1^2$.

STEP 2: Suppose (4) is true for some $n = k \geq 1$, that is

$$f_{2k} = f_{k+1}^2 - f_{k-1}^2$$

and $n = k + 1$, that is

$$f_{2k+2} = f_{k+2}^2 - f_k^2$$

STEP 3: Prove that (4) is true for $n = k + 2$, that is

$$f_{2k+4} \stackrel{?}{=} f_{k+3}^2 - f_{k+1}^2$$

We have

$$\begin{aligned} f_{2k+4} = f_{2k+3} + f_{2k+2} = f_{2k+2} + f_{2k+1} + f_{2k+2} &= 2f_{2k+2} + f_{2k+1} \\ &= 2f_{2k+2} + f_{2k+2} - f_{2k} \\ &= 3f_{2k+2} - f_{2k} \\ &\stackrel{\textbf{ST.2}}{=} 3(f_{k+2}^2 - f_k^2) - (f_{k+1}^2 - f_{k-1}^2) \\ &\stackrel{?}{=} f_{k+3}^2 - f_{k+1}^2 \end{aligned}$$

We now show that which is true, since

$$\begin{gathered} 3(f_{k+2}^2 - f_k^2) - (f_{k+1}^2 - f_{k-1}^2) \stackrel{?}{=} f_{k+3}^2 - f_{k+1}^2 \\ \Uparrow \\ 3(f_{k+2}^2 - f_k^2) - f_{k+1}^2 + f_{k-1}^2 \stackrel{?}{=} f_{k+3}^2 - f_{k+1}^2 \\ \Uparrow \\ 3(f_{k+2}^2 - f_k^2) + f_{k-1}^2 \stackrel{?}{=} f_{k+3}^2 \\ \Uparrow \\ 3(f_{k+2}^2 - f_k^2) \stackrel{?}{=} f_{k+3}^2 - f_{k-1}^2 \\ \Uparrow \\ 3(f_{k+2} - f_k)(f_{k+2} + f_k) \stackrel{?}{=} (f_{k+3} - f_{k-1})(f_{k+3} + f_{k-1}) \\ \Uparrow \\ 3f_{k+1}(f_{k+1} + f_k + f_k) \stackrel{?}{=} (f_{k+2} + f_{k+1} - f_{k-1})(f_{k+2} + f_{k+1} + f_{k-1}) \\ \Uparrow \\ 3f_{k+1}(f_{k+1} + 2f_k) \stackrel{?}{=} (f_{k+1} + f_k + f_{k+1} - f_{k-1})(f_{k+1} + f_k + f_{k+1} + f_{k-1}) \\ \Uparrow \\ 3f_{k+1}(f_{k+1} + 2f_k) = (f_{k+1} + f_k + f_k)(f_{k+1} + f_{k+1} + f_{k+1}) \end{gathered}$$

THEOREM 1: Let n be a positive integer ≥ 3 and let

$$\alpha = \frac{1+\sqrt{5}}{2}$$

Then

$$f_n > \alpha^{n-2} \tag{5}$$

Proof:

STEP 1: For $n = 3$ (5) is true, since

$$f_3 = 2 > \frac{1+\sqrt{5}}{2}$$

Similarly, for $n = 4$ (5) is true, since

$$f_4 = 3 > \left(\frac{1+\sqrt{5}}{2}\right)^2$$

STEP 2: Suppose (5) is true for some $n = k \geq 3$, that is

$$f_k > \alpha^{k-2}$$

and $n = k+1$, that is

$$f_{k+1} > \alpha^{k-1}$$

STEP 3: Prove that (5) is true for $n = k+2$, that is

$$f_{k+2} \overset{?}{>} \alpha^k$$

To this end, note that α is a solution of $x^2 - x - 1 = 0$, therefore

$$\alpha^2 = \alpha + 1$$

hence

$$\begin{aligned} \alpha^k = \alpha^2 \cdot \alpha^{k-2} = (\alpha+1)\alpha^{k-2} &= \alpha^{k-1} + \alpha^{k-2} \\ &\overset{\textbf{ST.2}}{<} f_{k+1} + f_k = f_{k+2} \end{aligned}$$

THEOREM 2: Let n be a positive integer and let

$$\alpha = \frac{1+\sqrt{5}}{2} \quad \text{and} \quad \beta = \frac{1-\sqrt{5}}{2}$$

Then

$$f_n = \frac{1}{\sqrt{5}}(\alpha^n - \beta^n) \tag{6}$$

Proof:

STEP 1: For $n = 1$ (6) is true, since

$$\frac{1}{\sqrt{5}}(\alpha^1 - \beta^1) = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2} - \frac{1-\sqrt{5}}{2}\right)$$

$$= \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}-1+\sqrt{5}}{2}\right) = \frac{1}{\sqrt{5}}\left(\frac{2\sqrt{5}}{2}\right) = \frac{1}{\sqrt{5}}\left(\sqrt{5}\right) = 1 = f_1$$

Similarly, for $n = 2$ (6) is true, since

$$\frac{1}{\sqrt{5}}(\alpha^2 - \beta^2) = \frac{1}{\sqrt{5}}\left(\left(\frac{1+\sqrt{5}}{2}\right)^2 - \left(\frac{1-\sqrt{5}}{2}\right)^2\right)$$

$$= \frac{1}{\sqrt{5}}\left(\frac{1^2+2\sqrt{5}+(\sqrt{5})^2}{4} - \frac{1^2-2\sqrt{5}+(\sqrt{5})^2}{4}\right)$$

$$= \frac{1}{\sqrt{5}}\left(\frac{1^2+2\sqrt{5}+(\sqrt{5})^2-1^2+2\sqrt{5}-(\sqrt{5})^2}{4}\right)$$

$$= \frac{1}{\sqrt{5}}\left(\frac{4\sqrt{5}}{4}\right) = \frac{1}{\sqrt{5}}\left(\sqrt{5}\right) = 1 = f_2$$

STEP 2: Suppose (6) is true for some $n = k \geq 1$, that is

$$f_k = \frac{1}{\sqrt{5}}(\alpha^k - \beta^k)$$

and $n = k+1$, that is

$$f_{k+1} = \frac{1}{\sqrt{5}}(\alpha^{k+1} - \beta^{k+1})$$

STEP 3: Prove that (6) is true for $n = k+2$, that is

$$f_{k+2} \stackrel{?}{=} \frac{1}{\sqrt{5}}(\alpha^{k+2} - \beta^{k+2})$$

To this end, note that α and β are solutions of $x^2 - x - 1 = 0$, therefore

$$\alpha^2 = \alpha + 1 \quad \text{and} \quad \beta^2 = \beta + 1$$

hence

$$\frac{1}{\sqrt{5}}(\alpha^{k+2} - \beta^{k+2}) = \frac{1}{\sqrt{5}}(\alpha^2\alpha^k - \beta^2\beta^k) = \frac{1}{\sqrt{5}}((\alpha+1)\alpha^k - (\beta+1)\beta^k)$$

$$= \frac{1}{\sqrt{5}}(\alpha^{k+1} + \alpha^k - \beta^{k+1} - \beta^k)$$

$$= \frac{1}{\sqrt{5}}(\alpha^{k+1} - \beta^{k+1} + \alpha^k - \beta^k)$$

$$= \frac{1}{\sqrt{5}}(\alpha^{k+1} - \beta^{k+1}) + \frac{1}{\sqrt{5}}(\alpha^k - \beta^k) \stackrel{\textbf{ST.2}}{=} f_{k+1} + f_k = f_{k+2}$$