Mathematical Induction

EXAMPLE 1: Prove that

$$1+3+5+\ldots+(2n-1)=n^2 \tag{$*$}$$

for any integer $n \geq 1$.

Proof:

STEP 1: For $n=1$ $(*)$ is true, since $1=1^2$.

STEP 2: Suppose $(*)$ is true for some $n=k\geq 1$, that is

$$1+3+5+\ldots+(2k-1)=k^2.$$

STEP 3: Prove that $(*)$ is true for $n=k+1$, that is

$$1+3+5+\ldots+(2k-1)+(2k+1)\overset{?}{=}(k+1)^2.$$

We have:

$$1+3+5+\ldots+(2k-1)+(2k+1)\overset{\textbf{ST.2}}{=}k^2+(2k+1)=(k+1)^2.$$

EXAMPLE 2: Prove that

$$1+2+3+\ldots+n=\frac{n(n+1)}{2} \tag{$*$}$$

for any integer $n \geq 1$.

Proof:

STEP 1: For $n=1$ $(*)$ is true, since

$$1=\frac{1(1+1)}{2}.$$

STEP 2: Suppose $(*)$ is true for some $n=k\geq 1$, that is

$$1+2+3+\ldots+k=\frac{k(k+1)}{2}.$$

STEP 3: Prove that $(*)$ is true for $n=k+1$, that is

$$1+2+3+\ldots+k+(k+1)\overset{?}{=}\frac{(k+1)(k+2)}{2}.$$

We have

$$1+2+3+\ldots+k+(k+1)\overset{\textbf{ST.2}}{=}\frac{k(k+1)}{2}+(k+1)\overset{?}{=}\frac{(k+1)(k+2)}{2},$$

which is true, since

$$\frac{k(k+1)}{2}+(k+1)\overset{?}{=}\frac{(k+1)(k+2)}{2}$$

$$\Uparrow$$

$$k(k+1)+2(k+1)\overset{?}{=}(k+1)(k+2)$$

$$\Uparrow$$

$$k^2+k+2k+2=k^2+3k+2.$$

EXAMPLE 3: Prove that

$$1^2+2^2+3^2+\ldots+n^2=\frac{n(n+1)(2n+1)}{6} \tag{$*$}$$

for any integer $n\geq 1$.

Proof:

STEP 1: For $n=1$ $(*)$ is true, since

$$1^2=\frac{1(1+1)(2\cdot 1+1)}{6}.$$

STEP 2: Suppose $(*)$ is true for some $n=k\geq 1$, that is

$$1^2+2^2+3^2+\ldots+k^2=\frac{k(k+1)(2k+1)}{6}.$$

STEP 3: Prove that $(*)$ is true for $n=k+1$, that is

$$1^2+2^2+3^2+\ldots+k^2+(k+1)^2\stackrel{?}{=}\frac{(k+1)(k+2)(2k+3)}{6}.$$

We have

$$1^2+2^2+3^2+\ldots+k^2+(k+1)^2\stackrel{\mathbf{ST.2}}{=}\frac{k(k+1)(2k+1)}{6}+(k+1)^2\stackrel{?}{=}\frac{(k+1)(k+2)(2k+3)}{6},$$

which is true, since

$$\begin{gathered}\frac{k(k+1)(2k+1)}{6}+(k+1)^2\stackrel{?}{=}\frac{(k+1)(k+2)(2k+3)}{6}\\ \Uparrow\\ k(k+1)(2k+1)+6(k+1)^2\stackrel{?}{=}(k+1)(k+2)(2k+3)\\ \Uparrow\\ (k^2+k)(2k+1)+6(k+1)^2\stackrel{?}{=}(k^2+3k+2)(2k+3)\\ \Uparrow\\ 2k^3+k^2+2k^2+k+6k^2+12k+6=2k^3+3k^2+6k^2+9k+4k+6.\end{gathered}$$

EXAMPLE 4: Prove that

$$\left(1-\frac{1}{2}\right)\left(1-\frac{1}{3}\right)\left(1-\frac{1}{4}\right)\ldots\left(1-\frac{1}{n}\right)=\frac{1}{n} \tag{$*$}$$

for any integer $n\geq 2$.

Proof:

STEP 1: For $n=2$ $(*)$ is true, since

$$\left(1-\frac{1}{2}\right)=\frac{1}{2}.$$

STEP 2: Suppose $(*)$ is true for some $n=k\geq 2$, that is

$$\left(1-\frac{1}{2}\right)\left(1-\frac{1}{3}\right)\left(1-\frac{1}{4}\right)\ldots\left(1-\frac{1}{k}\right)=\frac{1}{k}.$$

STEP 3: Prove that $(*)$ is true for $n = k+1$, that is

$$\left(1-\frac{1}{2}\right)\left(1-\frac{1}{3}\right)\left(1-\frac{1}{4}\right)\cdots\left(1-\frac{1}{k}\right)\left(1-\frac{1}{k+1}\right) \stackrel{?}{=} \frac{1}{k+1}.$$

We have:

$$\left(1-\frac{1}{2}\right)\left(1-\frac{1}{3}\right)\left(1-\frac{1}{4}\right)\cdots\left(1-\frac{1}{k}\right)\left(1-\frac{1}{k+1}\right) \stackrel{\textbf{ST.2}}{=} \frac{1}{k}\left(1-\frac{1}{k+1}\right) \stackrel{?}{=} \frac{1}{k+1},$$

Approach I: which is true, since

$$\begin{gathered}
\frac{1}{k}\left(1-\frac{1}{k+1}\right) \stackrel{?}{=} \frac{1}{k+1} \\
\Uparrow \\
\frac{1}{k}\left(\frac{k+1}{k+1}-\frac{1}{k+1}\right) \stackrel{?}{=} \frac{1}{k+1} \\
\Uparrow \\
\frac{1}{k}\left(\frac{k+1-1}{k+1}\right) \stackrel{?}{=} \frac{1}{k+1} \\
\Uparrow \\
\frac{1}{k}\left(\frac{k}{k+1}\right) \stackrel{?}{=} \frac{1}{k+1} \\
\Uparrow \\
\frac{k}{k(k+1)} = \frac{1}{k+1}.
\end{gathered}$$

Approach II: which is true, since

$$\begin{gathered}
\frac{1}{k}\left(1-\frac{1}{k+1}\right) \stackrel{?}{=} \frac{1}{k+1} \\
\Uparrow \\
\frac{k}{k}\left(1-\frac{1}{k+1}\right) \stackrel{?}{=} \frac{k}{k+1} \\
\Uparrow \\
1-\frac{1}{k+1} \stackrel{?}{=} \frac{k}{k+1} \\
\Uparrow \\
(k+1)-\frac{k+1}{k+1} \stackrel{?}{=} \frac{k(k+1)}{k+1} \\
\Uparrow \\
k+1-1 = k.
\end{gathered}$$

EXAMPLE 5: Prove that

$$\frac{1}{1\cdot 3}+\frac{1}{3\cdot 5}+\frac{1}{5\cdot 7}+\ldots+\frac{1}{(2n-1)(2n+1)}=\frac{n}{2n+1} \qquad (*)$$

for any integer $n\geq 1$.

Proof:

STEP 1: For $n=1$ $(*)$ is true, since

$$\frac{1}{1\cdot 3}=\frac{1}{2\cdot 1+1}.$$

STEP 2: Suppose $(*)$ is true for some $n=k\geq 1$, that is

$$\frac{1}{1\cdot 3}+\frac{1}{3\cdot 5}+\frac{1}{5\cdot 7}+\ldots+\frac{1}{(2k-1)(2k+1)}=\frac{k}{2k+1}.$$

STEP 3: Prove that $(*)$ is true for $n=k+1$, that is

$$\frac{1}{1\cdot 3}+\frac{1}{3\cdot 5}+\frac{1}{5\cdot 7}+\ldots+\frac{1}{(2k-1)(2k+1)}+\frac{1}{(2(k+1)-1)(2(k+1)+1)}\stackrel{?}{=}\frac{k+1}{2(k+1)+1}.$$

We have

$$\begin{aligned}
&\frac{1}{1\cdot 3}+\frac{1}{3\cdot 5}+\frac{1}{5\cdot 7}+\ldots+\frac{1}{(2k-1)(2k+1)}+\frac{1}{(2(k+1)-1)(2(k+1)+1)}\\
&=\frac{1}{1\cdot 3}+\frac{1}{3\cdot 5}+\frac{1}{5\cdot 7}+\ldots+\frac{1}{(2k-1)(2k+1)}+\frac{1}{(2k+2-1)(2k+2+1)}\\
&=\frac{1}{1\cdot 3}+\frac{1}{3\cdot 5}+\frac{1}{5\cdot 7}+\ldots+\frac{1}{(2k-1)(2k+1)}+\frac{1}{(2k+1)(2k+3)}\\
&\stackrel{\textbf{ST.2}}{=}\frac{k}{2k+1}+\frac{1}{(2k+1)(2k+3)}\stackrel{?}{=}\frac{k+1}{2(k+1)+1},
\end{aligned}$$

which is true, since

$$\frac{k}{2k+1}+\frac{1}{(2k+1)(2k+3)}\stackrel{?}{=}\frac{k+1}{2(k+1)+1}$$

$$\Uparrow$$

$$\frac{k}{2k+1}+\frac{1}{(2k+1)(2k+3)}\stackrel{?}{=}\frac{k+1}{2k+3}$$

$$\Uparrow$$

$$\frac{k(2k+3)}{(2k+1)(2k+3)}+\frac{1}{(2k+1)(2k+3)}\stackrel{?}{=}\frac{k+1}{2k+3}$$

$$\Uparrow$$

$$\frac{k(2k+3)+1}{(2k+1)(2k+3)}\stackrel{?}{=}\frac{k+1}{2k+3}$$

$$\Uparrow$$

$$\frac{k(2k+3)+1}{2k+1}\stackrel{?}{=}k+1$$

$$\Uparrow$$

$$k(2k+3)+1\stackrel{?}{=}(k+1)(2k+1)$$

$$\Uparrow$$

$$2k^2+3k+1=2k^2+k+2k+1.$$

EXAMPLE 6: Prove that

$$1 \cdot 2 + 2 \cdot 3 + 3 \cdot 4 + \ldots + (n-1)n = \frac{n(n-1)(n+1)}{3} \qquad (*)$$

for any integer $n \geq 2$.

Proof:

STEP 1: For $n = 2$ $(*)$ is true, since

$$1 \cdot 2 = \frac{2(2-1)(2+1)}{3}.$$

STEP 2: Suppose $(*)$ is true for some $n = k \geq 2$, that is

$$1 \cdot 2 + 2 \cdot 3 + 3 \cdot 4 + \ldots + (k-1)k = \frac{k(k-1)(k+1)}{3}.$$

STEP 3: Prove that $(*)$ is true for $n = k+1$, that is

$$1 \cdot 2 + 2 \cdot 3 + 3 \cdot 4 + \ldots + (k-1)k + k(k+1) \stackrel{?}{=} \frac{(k+1)k(k+2)}{3}.$$

We have

$$1 \cdot 2 + 2 \cdot 3 + 3 \cdot 4 + \ldots + (k-1)k + k(k+1) \stackrel{\textbf{ST.2}}{=} \frac{k(k-1)(k+1)}{3} + k(k+1) \stackrel{?}{=} \frac{(k+1)k(k+2)}{3},$$

which is true, since

$$\begin{gathered}
\frac{k(k-1)(k+1)}{3} + k(k+1) \stackrel{?}{=} \frac{(k+1)k(k+2)}{3} \\
\Uparrow \\
k(k-1)(k+1) + 3k(k+1) \stackrel{?}{=} (k+1)k(k+2) \\
\Uparrow \\
(k^2-k)(k+1) + 3k^2 + 3k \stackrel{?}{=} (k^2+k)(k+2) \\
\Uparrow \\
k^3 + k^2 - k^2 - k + 3k^2 + 3k = k^3 + 2k^2 + k^2 + 2k.
\end{gathered}$$

EXAMPLE 7: Prove that
$$3 \mid 4^n - 1 \tag{$*$}$$
for any integer $n \geq 1$.
Proof:
STEP 1: For $n = 1$ $(*)$ is true, since $3 \mid 4^1 - 1$.
STEP 2: Suppose $(*)$ is true for some $n = k \geq 1$, that is $3 \mid 4^k - 1$.
STEP 3: Prove that $(*)$ is true for $n = k + 1$, that is $3 \mid 4^{k+1} - 1$. We have
$$4^{k+1} - 1 = 4 \cdot 4^k - 1 = (3+1)4^k - 1 = \underbrace{3 \cdot 4^k}_{\text{div. by } 3} + \underbrace{4^k - 1}_{\substack{\text{St. } 2 \\ \text{div. by } 3}}.$$

EXAMPLE 8: Prove that
$$8 \mid 3^{2n} - 1 \tag{$*$}$$
for any integer $n \geq 1$.
Proof:
STEP 1: For $n = 1$ $(*)$ is true, since $8 \mid 3^{2\cdot 1} - 1$.
STEP 2: Suppose $(*)$ is true for some $n = k \geq 1$, that is $8 \mid 3^{2k} - 1$.
STEP 3: Prove that $(*)$ is true for $n = k + 1$, that is $8 \mid 3^{2(k+1)} - 1$. We have
$$3^{2(k+1)} - 1 = 3^{2k+2} - 1 = 3^{2k} \cdot 9 - 1 = 3^{2k}(8+1) - 1 = \underbrace{3^{2k} \cdot 8}_{\text{div. by } 8} + \underbrace{3^{2k} - 1}_{\substack{\text{St. } 2 \\ \text{div. by } 8}}.$$

EXAMPLE 9: Prove that
$$3 \mid n^3 - n \tag{$*$}$$
for any integer $n \geq 0$.
Proof:
STEP 1: For $n = 0$ $(*)$ is true, since $3 \mid 0^3 - 0$.
STEP 2: Suppose $(*)$ is true for some $n = k \geq 0$, that is $3 \mid k^3 - k$.
STEP 3: Prove that $(*)$ is true for $n = k + 1$, that is $3 \mid (k+1)^3 - (k+1)$. We have
$$(k+1)^3 - (k+1) = k^3 + 3k^2 + 3k + 1 - k - 1 = \underbrace{k^3 - k}_{\substack{\text{St. } 2 \\ \text{div. by } 3}} + \underbrace{3k^2 + 3k}_{\text{div. by } 3}.$$

EXAMPLE 10: Prove that
$$5 \mid n^5 - n \qquad (*)$$
for any integer $n \geq 0$.

Proof:

STEP 1: For $n = 0$ $(*)$ is true, since $5 \mid 0^5 - 0$.

STEP 2: Suppose $(*)$ is true for some $n = k \geq 0$, that is $5 \mid k^5 - k$.

STEP 3: Prove that $(*)$ is true for $n = k + 1$, that is $5 \mid (k+1)^5 - (k+1)$. We have
$$(k+1)^5 - (k+1) = k^5 + 5k^4 + 10k^3 + 10k^2 + 5k + 1 - k - 1 = \underbrace{k^5 - k}_{\substack{\text{St. 2} \\ \text{div. by 5}}} + \underbrace{5k^4 + 10k^3 + 10k^2 + 5k}_{\text{div. by 5}}.$$

EXAMPLE 11: Prove that
$$3 \mid n^3 - 7n + 3 \qquad (*)$$
for any integer $n \geq 0$.

Proof:

STEP 1: For $n = 0$ $(*)$ is true, since $3 \mid 0^3 - 7 \cdot 0 + 3$.

STEP 2: Suppose $(*)$ is true for some $n = k \geq 0$, that is $3 \mid k^3 - 7k + 3$.

STEP 3: Prove that $(*)$ is true for $n = k + 1$, that is $3 \mid (k+1)^3 - 7(k+1) + 3$. We have
$$(k+1)^3 - 7(k+1) + 3 = k^3 + 3k^2 + 3k + 1 - 7k - 7 + 3 = \underbrace{k^3 - 7k + 3}_{\substack{\text{St. 2} \\ \text{div. by 3}}} + \underbrace{3k^2 + 3k - 6}_{\text{div. by 3}}.$$

EXAMPLE 12: Prove that
$$7 \mid n^7 - n \qquad (*)$$
for any integer $n \geq 1$.

Proof:

STEP 1: For $n = 1$ $(*)$ is true, since $7 \mid 1^7 - 1$.

STEP 2: Suppose $(*)$ is true for some $n = k \geq 1$, that is $7 \mid k^7 - k$.

STEP 3: Prove that $(*)$ is true for $n = k + 1$, that is $7 \mid (k+1)^7 - (k+1)$. We have
$$(k+1)^7 - (k+1) = k^7 + 7k^6 + 21k^5 + 35k^4 + 35k^3 + 21k^2 + 7k + 1 - k - 1$$
$$= \underbrace{k^7 - k}_{\substack{\text{St. 2} \\ \text{div. by 7}}} + \underbrace{7k^6 + 21k^5 + 35k^4 + 35k^3 + 21k^2 + 7k}_{\text{div. by 7}}.$$

EXAMPLE 13: Prove that

$$2^n > n \tag{$*$}$$

for any integer $n \geq 1$.

Proof:

STEP 1: For $n = 1$ $(*)$ is true, since $2^1 > 1$.

STEP 2: Suppose $(*)$ is true for some $n = k \geq 1$, that is $2^k > k$.

STEP 3: Prove that $(*)$ is true for $n = k + 1$, that is $2^{k+1} \stackrel{?}{>} k + 1$. We have

$$2^{k+1} = 2 \cdot 2^k \stackrel{\textbf{ST.2}}{>} 2k \stackrel{?}{\geq} k + 1,$$

which is true, since $k \geq 1$.

EXAMPLE 14: Prove that

$$2^n < n! \tag{$*$}$$

for any integer $n \geq 4$.

Proof:

STEP 1: For $n = 4$ $(*)$ is true, since $2^4 < 4!$ $(16 < 24)$.

STEP 2: Suppose $(*)$ is true for some $n = k \geq 4$, that is $2^k < k!$.

STEP 3: Prove that $(*)$ is true for $n = k + 1$, that is $2^{k+1} \stackrel{?}{<} (k + 1)!$. We have

$$2^{k+1} = 2 \cdot 2^k \stackrel{\textbf{ST.2}}{<} 2k! \stackrel{?}{<} (k + 1)!,$$

which is true, since

$$\begin{gathered} 2k! \stackrel{?}{<} (k+1)! \\ \Uparrow \\ 2k! \stackrel{?}{<} k!(k+1) \\ \Uparrow \\ 2 \stackrel{?}{<} k+1 \\ \Uparrow \\ 1 < k. \end{gathered}$$

EXAMPLE 15: Prove that

$$n! \leq n^n \tag{$*$}$$

for any integer $n \geq 1$.

Proof:

STEP 1: For $n = 1$ $(*)$ is true, since $1! = 1^1$.

STEP 2: Suppose $(*)$ is true for some $n = k \geq 1$, that is $k! \leq k^k$.

STEP 3: Prove that $(*)$ is true for $n = k + 1$, that is $(k + 1)! \stackrel{?}{\leq} (k + 1)^{k+1}$. We have

$$(k + 1)! = k! \cdot (k + 1) \stackrel{\textbf{ST.2}}{\leq} k^k \cdot (k + 1) \stackrel{?}{\leq} (k + 1)^{k+1},$$

which is true, since $k^k < (k + 1)^k$.

EXAMPLE 16: Prove that

$$3^n < n! \tag{$*$}$$

for any integer $n \geq 7$.

Proof:

STEP 1: For $n = 7$ $(*)$ is true, since $3^7 < 7!$ $(2187 < 5040)$.

STEP 2: Suppose $(*)$ is true for some $n = k \geq 7$, that is $3^k < k!$.

STEP 3: Prove that $(*)$ is true for $n = k+1$, that is $3^{k+1} \stackrel{?}{<} (k+1)!$. We have

$$3^{k+1} = 3 \cdot 3^k \stackrel{\textbf{ST.2}}{<} 3k! \stackrel{?}{<} (k+1)!,$$

which is true, since

$$\begin{gathered} 3k! \stackrel{?}{<} (k+1)! \\ \Uparrow \\ 3k! \stackrel{?}{<} k!(k+1) \\ \Uparrow \\ 3 \stackrel{?}{<} k+1 \\ \Uparrow \\ 2 < k. \end{gathered}$$

EXAMPLE 17: Prove that

$$3^n \geq 2n+1 \tag{$*$}$$

for any integer $n \geq 1$.

Proof:

STEP 1: For $n = 1$ $(*)$ is true, since $3^1 = 2 \cdot 1 + 1$.

STEP 2: Suppose $(*)$ is true for some $n = k \geq 1$, that is $3^k \geq 2k+1$.

STEP 3: Prove that $(*)$ is true for $n = k+1$, that is $3^{k+1} \stackrel{?}{\geq} 2(k+1)+1$. We have

$$3^{k+1} = 3 \cdot 3^k \stackrel{\textbf{ST.2}}{\geq} 3(2k+1) \stackrel{?}{\geq} 2(k+1)+1,$$

which is true, since

$$\begin{gathered} 3(2k+1) \stackrel{?}{\geq} 2(k+1)+1 \\ \Uparrow \\ 6k+3 \stackrel{?}{\geq} 2k+2+1 \\ \Uparrow \\ 4k \geq 0. \end{gathered}$$

EXAMPLE 18: Prove that

$$2^{n+2} \geq 2n+5 \tag{$*$}$$

for any integer $n \geq 1$.

Proof:

STEP 1: For $n = 1$ $(*)$ is true, since $2^{1+2} \geq 2 \cdot 1 + 5$.

STEP 2: Suppose $(*)$ is true for some $n = k \geq 1$, that is $2^{k+2} \geq 2k+5$.

STEP 3: Prove that $(*)$ is true for $n = k+1$, that is $2^{k+3} \overset{?}{\geq} 2(k+1)+5$. We have

$$2^{k+3} = 2 \cdot 2^{k+2} \overset{\textbf{ST.2}}{\geq} 3(2k+5) \overset{?}{\geq} 2(k+1)+5,$$

which is true, since

$$\begin{gathered} 3(2k+5) \overset{?}{\geq} 2(k+1)+5 \\ \Uparrow \\ 6k+15 \overset{?}{\geq} 2k+2+5 \\ \Uparrow \\ 4k+8 \geq 0. \end{gathered}$$

I. Prove by induction the following identities:

1^*. $$\frac{1}{\sqrt{1}+\sqrt{2}} + \frac{1}{\sqrt{2}+\sqrt{3}} + \frac{1}{\sqrt{3}+\sqrt{4}} + \ldots + \frac{1}{\sqrt{n-1}+\sqrt{n}} = \sqrt{n} - 1.$$

2^{**}. $$\sin\varphi + \sin 2\varphi + \sin 3\varphi + \ldots + \sin n\varphi = \frac{\sin\frac{n\varphi}{2}\sin\frac{(n+1)\varphi}{2}}{\sin\frac{\varphi}{2}}.$$

II. Prove by induction the following inequalities:

1^*. $(2n)! < 2^{2n}(n!)^2$ for any integer $n \geq 1$.

2^{**}. $(n+1)^n < n^{n+1}$ for any integer $n \geq 3$.

3^{**}. $\dfrac{a_1^n + a_2^n}{2} \geq \left(\dfrac{a_1+a_2}{2}\right)^n$ for any positive numbers a_1, a_2 and for any integer $n \geq 1$.

4^{**}. $\dfrac{1}{1^2} + \dfrac{1}{2^2} + \dfrac{1}{3^2} + \ldots + \dfrac{1}{n^2} < 2$ for any integer $n \geq 1$.

III. Prove by induction the following problems:

1^*. $3^{2n+3} + 40n - 27$ is divisible by 64 for any nonnegative integer n.

2^*. $5^{2n+1} \cdot 2^{n+2} + 3^{n+2} \cdot 2^{2n+1}$ is divisible by 19 for any nonnegative integer n.

3^{**}. $\dfrac{1}{\sqrt{5}}\left(\left(\dfrac{1+\sqrt{5}}{2}\right)^n - \left(\dfrac{1-\sqrt{5}}{2}\right)^n\right)$ is integer for any nonnegative integer n.