Sums and Products

DEFINITION:

$$\sum_{k=m}^{n} a_k = a_m + a_{m+1} + \ldots + a_n$$

EXAMPLES:

(a) $\displaystyle\sum_{k=1}^{4} a_k = a_1 + a_2 + a_3 + a_4$

(b) $\displaystyle\sum_{k=3}^{7} b_k = b_3 + b_4 + b_5 + b_6 + b_7$

(c) $\displaystyle\sum_{k=-3}^{7} b_k = b_{-3} + b_{-2} + b_{-1} + b_0 + b_1 + b_2 + b_3 + b_4 + b_5 + b_6 + b_7$

(d) $\displaystyle\sum_{k=1}^{5} 2 = 2 + 2 + 2 + 2 + 2 = 10$

(e) $\displaystyle\sum_{k=3}^{5} (-2) = (-2) + (-2) + (-2) = -6$

(f) $\displaystyle\sum_{k=1}^{4} k = 1 + 2 + 3 + 4 = 10$

(g) $\displaystyle\sum_{k=1}^{3} (k-1)^2 = 0^2 + 1^2 + 2^2 = 0 + 1 + 4 = 5$

(h) $\displaystyle\sum_{k=3}^{7} (2k^2 + 1) = 19 + 33 + 51 + 73 + 99 = 275$

(i) $\displaystyle\sum_{k=3}^{7} (-1)^k (2k^2 + 1) = -19 + 33 - 51 + 73 - 99 = -63$

(j) $\displaystyle\sum_{k=2}^{4} \frac{1}{2^k} = \frac{1}{2^2} + \frac{1}{2^3} + \frac{1}{2^4} = \frac{1}{4} + \frac{1}{8} + \frac{1}{16} = \frac{7}{16}$

EXAMPLE: Write

$$\frac{2}{3} + \frac{3}{4} + \frac{4}{5} + \frac{5}{6}$$

using summation notation.

EXAMPLE: Write

$$\frac{2}{3}+\frac{3}{4}+\frac{4}{5}+\frac{5}{6}$$

using summation notation.

Solution: We have

$$\frac{2}{3}+\frac{3}{4}+\frac{4}{5}+\frac{5}{6}=\sum_{k=1}^{4}\frac{k+1}{k+2}=\sum_{k=2}^{5}\frac{k}{k+1}=\sum_{k=0}^{3}\frac{k+2}{k+3}$$

EXAMPLE: Write

$$3^4+4^4+5^4+6^4+7^4+8^4$$

using summation notation.

Solution: We have

$$3^4+4^4+5^4+6^4+7^4+8^4=\sum_{k=3}^{8}k^4=\sum_{k=1}^{6}(k+2)^4$$

EXAMPLE: Write

$$3^5-4^5+5^5-6^5+7^5-8^5$$

using summation notation.

Solution: We have

$$3^5-4^5+5^5-6^5+7^5-8^5=\sum_{k=3}^{8}(-1)^{k+1}k^5=\sum_{k=2}^{7}(-1)^{k}(k+1)^5=\sum_{k=1}^{6}(-1)^{k+1}(k+2)^5$$

EXAMPLE: Write

$$a+a^2+a^3+a^4+a^5+a^6$$

using summation notation.

Solution: We have

$$a+a^2+a^3+a^4+a^5+a^6=\sum_{k=1}^{6}a^k$$

EXAMPLE: Write

$$a-a^2+a^3-a^4+a^5-a^6$$

using summation notation.

EXAMPLE: Write

$$a - a^2 + a^3 - a^4 + a^5 - a^6$$

using summation notation.

Solution: We have

$$a - a^2 + a^3 - a^4 + a^5 - a^6 = \sum_{k=1}^{6} (-1)^{k+1} a^k$$

EXAMPLE: Write

$$1 - a + a^2 - a^3 + a^4 - a^5$$

using summation notation.

Solution: We have

$$1 - a + a^2 - a^3 + a^4 - a^5 = \sum_{k=0}^{5} (-1)^k a^k = \sum_{k=1}^{6} (-1)^{k+1} a^{k-1}$$

EXAMPLE: Write

$$\frac{50}{10 \cdot 19} + \frac{65}{11 \cdot 21} + \frac{82}{12 \cdot 23} + \frac{101}{13 \cdot 25}$$

using summation notation.

Solution: We have

$$\frac{50}{10 \cdot 19} + \frac{65}{11 \cdot 21} + \frac{82}{12 \cdot 23} + \frac{101}{13 \cdot 25} = \sum_{k=7}^{10} \frac{k^2 + 1}{(k+3)(2k+5)}$$

DEFINITION:

$$\prod_{k=m}^{n} a_k = a_m a_{m+1} \ldots a_n$$

EXAMPLES:

(a) $\displaystyle\prod_{k=1}^{5} k = 1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 = 5! = 120$

(b) $\displaystyle\prod_{k=3}^{7} 2 = 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 = 2^5 = 32$

(c) $\displaystyle\prod_{k=4}^{8} 2^{k-3} = 2 \cdot 2^2 \cdot 2^3 \cdot 2^4 \cdot 2^5 = 2^{1+2+3+4+5} = 2^{15} = 32768$

Geometric Series

DEFINITION: A geometric series is a series with a constant ratio between successive terms.

$$\sum_{k=1}^{\infty} ar^{k-1} = a + ar + ar^2 + \ldots + ar^k + \ldots \quad (a \neq 0)$$

THEOREM: For $r \neq 1$, the sum of the first n terms of a geometric series is

$$\sum_{k=1}^{n} ar^{k-1} = a + ar + ar^2 + \ldots + ar^{n-1} = \frac{a(1-r^n)}{1-r}$$

Proof: We have

$$s_n = a + ar + ar^2 + \ldots + ar^{n-1} \quad \text{and} \quad rs_n = ar + ar^2 + ar^3 + \ldots + ar^n$$

Subtracting these equations, we get

$$s_n - rs_n = a - ar^n \quad \Longrightarrow \quad s_n(1-r) = a(1-r^n) \quad \Longrightarrow \quad s_n = \frac{a(1-r^n)}{1-r}$$

EXAMPLE: Find the sum of the geometric series

$$\sum_{k=1}^{5} \left(\frac{1}{2}\right)^{k-1}$$

Solution: We have

$$\sum_{k=1}^{5} \left(\frac{1}{2}\right)^{k-1} = \left[a = 1, \ r = \frac{1}{2}\right] = \frac{1 - \left(\frac{1}{2}\right)^5}{1 - \frac{1}{2}} = \frac{31}{16}$$

EXAMPLE: Find the sum of the geometric series

$$\sum_{k=1}^{5} \left(\frac{1}{2}\right)^{k}$$

Solution: We have

$$\sum_{k=1}^{5} \left(\frac{1}{2}\right)^{k} = \sum_{k=1}^{5} \frac{1}{2} \cdot \left(\frac{1}{2}\right)^{k-1} = \left[a = \frac{1}{2}, \ r = \frac{1}{2}\right] = \frac{1}{2} \cdot \frac{1 - \left(\frac{1}{2}\right)^5}{1 - \frac{1}{2}} = \frac{31}{32}$$

Telescoping Sums

EXAMPLE: Evaluate
$$\frac{1}{\sqrt{1}+\sqrt{2}}+\frac{1}{\sqrt{2}+\sqrt{3}}+\frac{1}{\sqrt{3}+\sqrt{4}}$$

Solution: We have
$$\frac{1}{\sqrt{1}+\sqrt{2}}+\frac{1}{\sqrt{2}+\sqrt{3}}+\frac{1}{\sqrt{3}+\sqrt{4}}=\frac{1}{\sqrt{2}+\sqrt{1}}+\frac{1}{\sqrt{3}+\sqrt{2}}+\frac{1}{\sqrt{4}+\sqrt{3}}$$
$$=\frac{\sqrt{2}-\sqrt{1}}{(\sqrt{2}+\sqrt{1})(\sqrt{2}-\sqrt{1})}+\frac{\sqrt{3}-\sqrt{2}}{(\sqrt{3}+\sqrt{2})(\sqrt{3}-\sqrt{2})}+\frac{\sqrt{4}-\sqrt{3}}{(\sqrt{4}+\sqrt{3})(\sqrt{4}-\sqrt{3})}$$
$$=\frac{\sqrt{2}-\sqrt{1}}{(\sqrt{2})^2-(\sqrt{1})^2}+\frac{\sqrt{3}-\sqrt{2}}{(\sqrt{3})^2-(\sqrt{2})^2}+\frac{\sqrt{4}-\sqrt{3}}{(\sqrt{4})^2-(\sqrt{3})^2}$$
$$=\frac{\sqrt{2}-\sqrt{1}}{2-1}+\frac{\sqrt{3}-\sqrt{2}}{3-2}+\frac{\sqrt{4}-\sqrt{3}}{4-3}=\frac{\sqrt{2}-\sqrt{1}}{1}+\frac{\sqrt{3}-\sqrt{2}}{1}+\frac{\sqrt{4}-\sqrt{3}}{1}$$
$$=\sqrt{2}-\sqrt{1}+\sqrt{3}-\sqrt{2}+\sqrt{4}-\sqrt{3}=-\sqrt{1}+\sqrt{4}=1$$

REMARK: In the same way one can prove that
$$\sum_{k=1}^{n-1}\frac{1}{\sqrt{k}+\sqrt{k+1}}=\sqrt{n}-1$$

EXAMPLE: Find
$$\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\frac{1}{3\cdot 4}+\frac{1}{4\cdot 5}+\dots$$

Solution: We have
$$s_n=\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\frac{1}{3\cdot 4}+\frac{1}{4\cdot 5}+\dots+\frac{1}{n(n+1)}\qquad\left[\frac{1}{k(k+1)}=\frac{1}{k}-\frac{1}{k+1}\right]$$
$$=\left(1-\frac{1}{2}\right)+\left(\frac{1}{2}-\frac{1}{3}\right)+\left(\frac{1}{3}-\frac{1}{4}\right)+\left(\frac{1}{4}-\frac{1}{5}\right)+\dots+\left(\frac{1}{n}-\frac{1}{n+1}\right)$$
$$=1+\left(-\frac{1}{2}+\frac{1}{2}\right)+\left(-\frac{1}{3}+\frac{1}{3}\right)+\left(-\frac{1}{4}+\frac{1}{4}\right)+\dots+\left(-\frac{1}{n}+\frac{1}{n}\right)-\frac{1}{n+1}$$
$$=1-\frac{1}{n+1}$$

Therefore
$$\sum_{k=1}^{\infty}\frac{1}{k(k+1)}=\lim_{n\to\infty}s_n=\lim_{n\to\infty}\left(1-\frac{1}{n+1}\right)=1$$

EXAMPLE: Show that
$$\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\dots+\frac{1}{n^2}<1$$

for any integer $n\geq 2$.

EXAMPLE: Show that

$$\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\ldots+\frac{1}{n^2}<1$$

for any integer $n \geq 2$.

Solution: We have

$$\begin{aligned}\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\ldots+\frac{1}{n^2} &< \frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\frac{1}{3\cdot 4}+\ldots+\frac{1}{(n-1)n}\\
&= \left(1-\frac{1}{2}\right)+\left(\frac{1}{2}-\frac{1}{3}\right)+\left(\frac{1}{3}-\frac{1}{4}\right)+\left(\frac{1}{4}-\frac{1}{5}\right)+\ldots+\left(\frac{1}{n-1}-\frac{1}{n}\right)\\
&= 1+\left(-\frac{1}{2}+\frac{1}{2}\right)+\left(-\frac{1}{3}+\frac{1}{3}\right)+\left(-\frac{1}{4}+\frac{1}{4}\right)+\ldots+\left(-\frac{1}{n-1}+\frac{1}{n-1}\right)-\frac{1}{n}\\
&= 1-\frac{1}{n}<1\end{aligned}$$